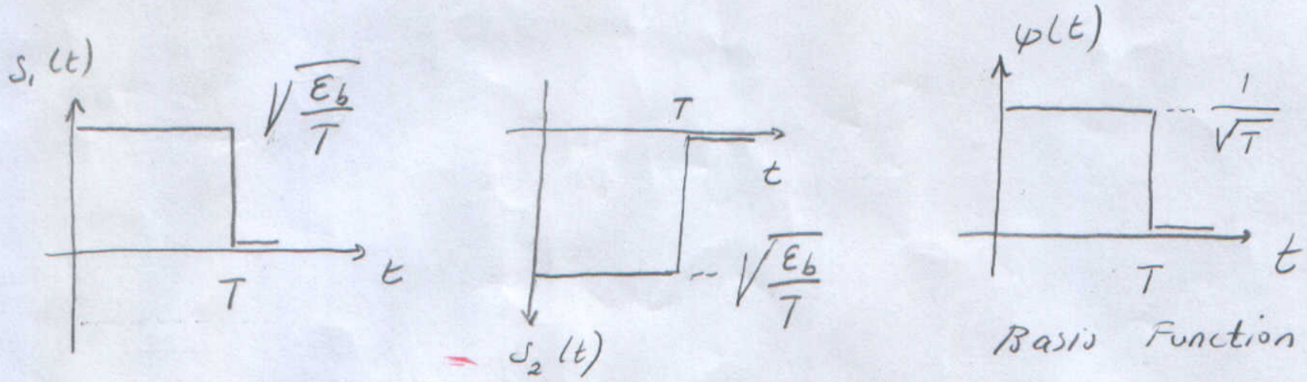
ECE 376

Formulating Probability of Error for Binary PAM

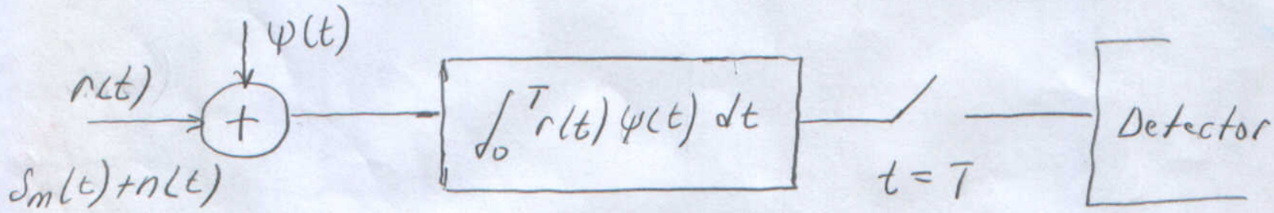
(1)



$$s_1(t) = \sqrt{\frac{\mathcal{E}_b}{T}} \quad 0 < t < T, \quad s_2(t) = -\sqrt{\frac{\mathcal{E}_b}{T}} \quad 0 < t < T$$

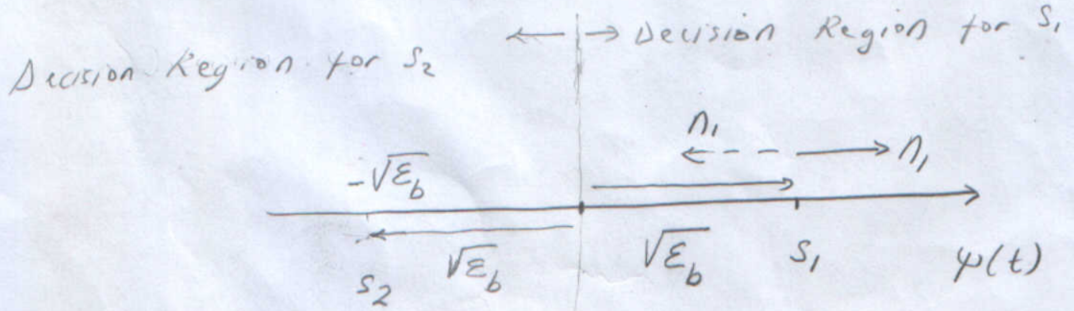
$$s_1 = [\, \mathcal{E}_b^{1/2} \,] \qquad s_2 = [\, -\mathcal{E}_b^{1/2} \,] \quad , \quad \mathcal{E}_b = \int_0^T s_m^2(t)\,dt$$

Block diagramme of Matched Filter Correlator



Assume s_1 was transmitted

$$\int_0^T r(t)\varphi(t)\,dt = \sqrt{\mathcal{E}_b} + n_1 = r \text{ (vector)}$$



Since it is one dimensional space n_1 will always be along $\psi(t)$ axis. Assuming that $s_1(t)$ and $s_2(t)$ have the representation above having equal energy.

Thus the threshold of decision is set to zero

Then error will result if $n_1 \leq -\mathcal{E}_b^{1/2}$

The same result could be obtained via $C(r, s_m)$ metric. In the above case of s_1 being transmitted

$$C(r, s_1) = 2r \cdot s_m - \|s_m\|^2$$

$$= 2[\sqrt{\mathcal{E}_b} + n_1] \cdot [\mathcal{E}_b^{1/2}] - \mathcal{E}_b$$

$$= \mathcal{E}_b + 2n_1 \mathcal{E}_b^{1/2}$$

$$C(r, s_2) = 2[\mathcal{E}_b^{1/2} + n_1][-\mathcal{E}_b^{1/2}] - \mathcal{E}_b$$

$$= -3\mathcal{E}_b - 2n_1 \mathcal{E}_b^{1/2}$$

For error to occur $C(r, s_2) > C(r, s_1)$ or

$$-3\mathcal{E}_b - 2n_1 \mathcal{E}_b^{1/2} > \mathcal{E}_b + 2n_1 \mathcal{E}_b^{1/2}$$

$$n_1 < -\mathcal{E}_b^{1/2}$$

Note that due to symmetry of Gaussian distribution

The definition of P_e is based on Eq (7.5.55)

that is

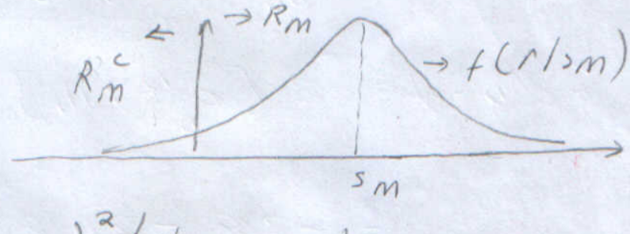
$$p(e/s_m) = \int_{R_m^c} f(r/s_m)\, dr$$

R_m means the region of correct decision

R_m^c means the region outside the region of

correct decision, i.e it the complimentary to

R_c



Note that

$$f(r/s_m) = \frac{1}{(\pi N_0)^{1/2}}\, e^{-(r-s_m)^2/N_0} \quad , \text{hence}$$

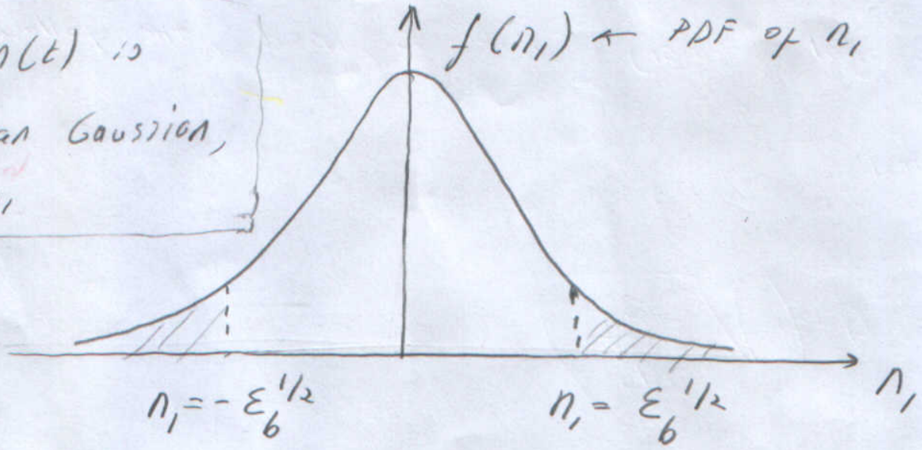
$$\int_{-\infty}^{0} \frac{1}{(\pi N_0)^{1/2}}\, e^{-(r-\overset{\sqrt{\varepsilon_b}}{s_m})^2/N_0}\, dr = \frac{1}{(\pi N_0)^{1/2}} \int_{-\infty}^{-\sqrt{\varepsilon_b}} e^{-x^2/N_0}\, dx$$

Same as n_1 integral

$-\infty < n_1 < -\mathcal{E}_b^{1/2}$ and $\mathcal{E}_b^{1/2} < n_1 < \infty$

will give the same area (see the figure below)

Since $n(t)$ is zero mean Gaussian, so is n_1



$$f(n_1) = \frac{1}{(\pi N_0)^{1/2}} e^{-n_1^2/N_0}$$

$$P_e = \text{Prob}\left(\mathcal{E}_b^{1/2} < n_1 < \infty\right) \leftarrow (\text{This is } P(e/s_2)$$

$$= \int_{\sqrt{\mathcal{E}_b}}^{\infty} \frac{1}{(\pi N_0)^{1/2}} e^{-n_1^2/N_0} \, dn_1$$

Make the substitution $\dfrac{n_1^2}{N_0} = \dfrac{x^2}{2}$

$$P_e(e/s, \text{binary PAM}) = \frac{1}{(2\pi)^{1/2}} \int_{\sqrt{\frac{2\mathcal{E}_b}{N_0}}}^{\infty} e^{-x^2/2} \, dx$$

$P_e = Q\left(\sqrt{\dfrac{2\mathcal{E}_b}{N_0}}\right)$ same as Eq (7.6.8) of Proakis

The above is the same as binary PSK with antipodal signalling

If othogonal binary PSK is used, the following is obtained

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

hence P_e (othogonal binary PSK)

$$>$$

P_e (antipodal binary PSK)

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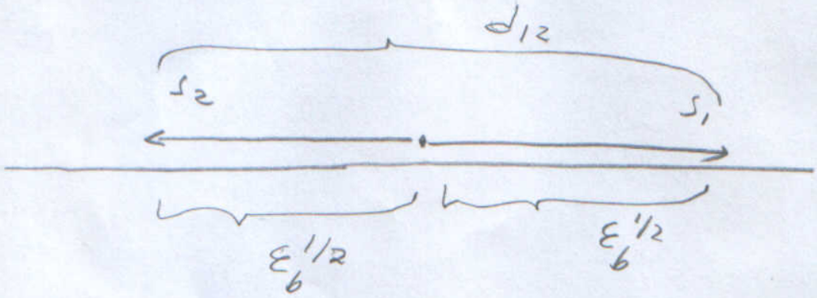
Addendum for orthogonal binary PSK (1a)

Note that the above $P_e = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$ is also valid for binary PSK where $s_1(t)$ and $s_2(t)$ are antipodal signals, i.e.,



Of course $s_1(t)$ and $s_2(t)$ must be equiprobable.

We observe that in Q function $\mathcal{E}_b$ may also be expressed in terms of the distance between end of vectors s_1 and s_2. In this manner

$$d_{12} = \sqrt{\mathcal{E}_b} + \sqrt{\mathcal{E}_b} = 2\sqrt{\mathcal{E}_b},$$

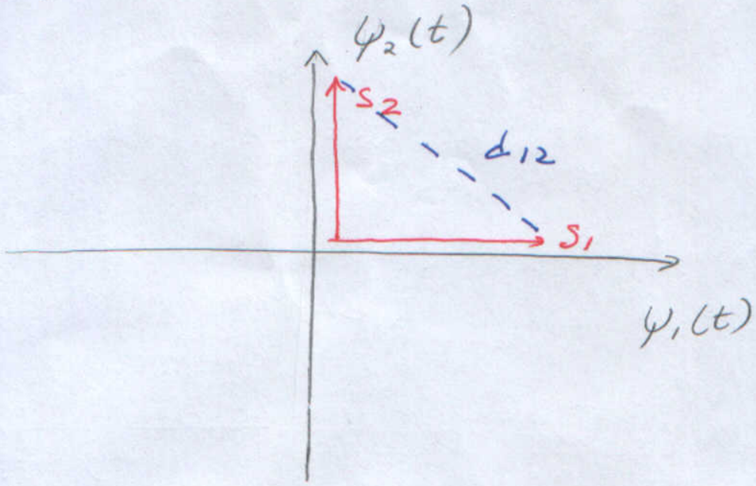
$$2\mathcal{E}_b = d_{12}^2/2 \quad \text{or} \quad d_{12}^2 = 4\mathcal{E}_b$$

Hence P_e may be written as

$$P_e = Q\left(\sqrt{\frac{\overset{2\mathcal{E}_b}{d_{12}^2}}{2N_0}}\right)$$

See also Proakis 2000 pp. 406-407

Therefore by taking the binary orthogonal PSK as follows



d_{12} (for orthogonal binary PSK) $= \sqrt{2\mathcal{E}_b}$

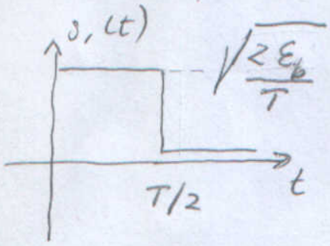
or $\quad d_{12}^2 = 2\,\mathcal{E}_b$, thus

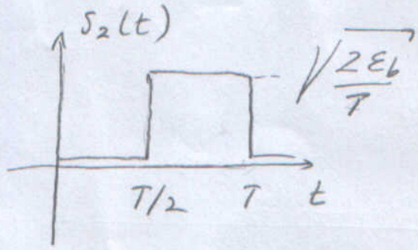
P_e (for orthogonal binary PSK) $= Q\left(\sqrt{\dfrac{\mathcal{E}_b}{N_0}}\right)$

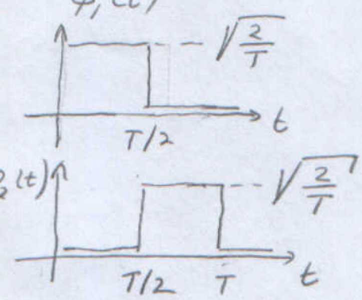
As expected P_e of orthogonal case is has larger values than P_e of antipodal case.

The above result may have been acheived by examining the correlation metrics values

For this case, the waveform illustrations are reproduced below







so $s_1(t) = \sqrt{\mathcal{E}_b}\,\varphi_1(t) + 0\,\varphi_2(t)$ $s_1 = [\sqrt{\mathcal{E}_b}\,; 0]$

$s_2(t) = 0\,\varphi_1(t) + \sqrt{\mathcal{E}_b}\,\varphi_2(t)$ $s_2 = [0; \sqrt{\mathcal{E}_b}\,]$

This way the OP from the correlators (or MF)

if s_1 was transmitted

$r_1 = \sqrt{\mathcal{E}_b} + n_1$, $r_2 = n_2$ or $r = [\sqrt{\mathcal{E}_b} + n_1, n_2]$

Normally test against correlator metrics for s_1 and s_2 should yield that $C(r, s_1) > C(r, s_2)$

So in case of error the opposite should occur, that is $C(r, s_2) > C(r, s_1)$

This is shown below

$$C(r, s_1) = 2r \cdot s_1 - \|s_1\|^2$$

$$= 2\left[\sqrt{\mathcal{E}_b} + n_1, n_2\right]\begin{bmatrix}\sqrt{\mathcal{E}_b} \\ 0\end{bmatrix} - \mathcal{E}_b$$

$$= 2\left(\mathcal{E}_b + \mathcal{E}_b^{1/2} n_1\right) - \mathcal{E}_b$$

$$C(r, s_2) = 2 r s_2 - \|s_2\|^2$$

$$= 2\left[\sqrt{\mathcal{E}_b} + n_1, n_2\right]\begin{bmatrix}0 \\ \sqrt{\mathcal{E}_b}\end{bmatrix} - \mathcal{E}_b$$

$$= 2 n_2 \mathcal{E}_b^{1/2} - \mathcal{E}_b$$

The condition of $C(r, s_2) \geq C(r, s_1)$

$$\sqrt{\mathcal{E}_b}\,(2n_2 - 1) \geq \sqrt{\mathcal{E}_b}\,(2\sqrt{\mathcal{E}_b} + 2n_1 - 1)$$

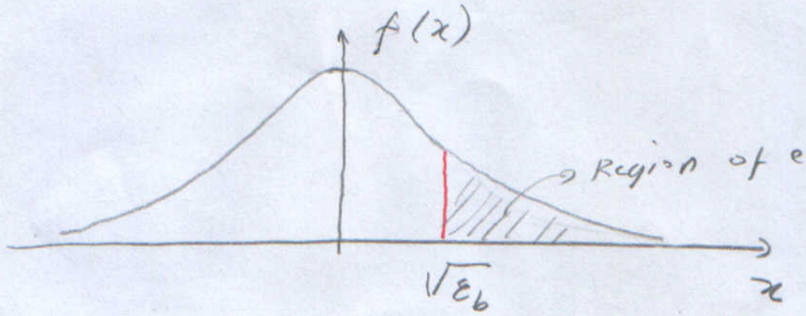
$$2n_2 - 2n_1 \geq 2\sqrt{\mathcal{E}_b}$$

$$n_2 - n_1 \geq \mathcal{E}_b^{1/2}$$

n_1 and n_2 are independent Gaussian variables with zero mean and each with $N_0/2$, hence $x = n_2 - n_1$ is again with zero mean, but with N_0 variance $(N_0/2 + N_0/2 = N_0)$

Hence

$$f(n_2 - n_1 = x) = \frac{1}{\sqrt{2\pi N_0}} e^{-x^2/2N_0}$$



Pe (of orthogonal binary PSK) = f(x) being larger than $\sqrt{E}$

$$= \frac{1}{\sqrt{2\pi N_0}} \int_{\sqrt{E_b}}^{\infty} e^{-x^2/2N_0}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{E_b/N_0}}^{\infty} e^{-x^2} dx$$

$$= Q\left[\sqrt{\frac{E_b}{N_0}}\right]$$

The same result as before