Çankaya University – ECE Department – ECE 376 (MT)

Student Name : Date : 05.04.2016
Student Number : Open Source Exam

Questions

1. (70 Points) A constellation is given in Fig. 1.1.

   a. Identify the type of modulation and dimensionality in this constellation. Write and plot the mathematical expression for the basis functions, $\psi_1(t)$, $\psi_2(t)$, write for the signal vectors $\mathbf{s_1} \cdots \mathbf{s_8}$, write and plot the corresponding signal waveforms of $s_1(t) \cdots s_8(t)$. Find the distance between signal vector ends. Determine the total energy used if all signals are sent from the transmitter with equal probability.
   b. Find the error and correct decision regions via the evaluations of correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$ for $\mathbf{s}_1$ and $\mathbf{s}_2$. Comment on how the probability error of the rest of the signals will be related to that of $\mathbf{s}_1$ and $\mathbf{s}_2$.

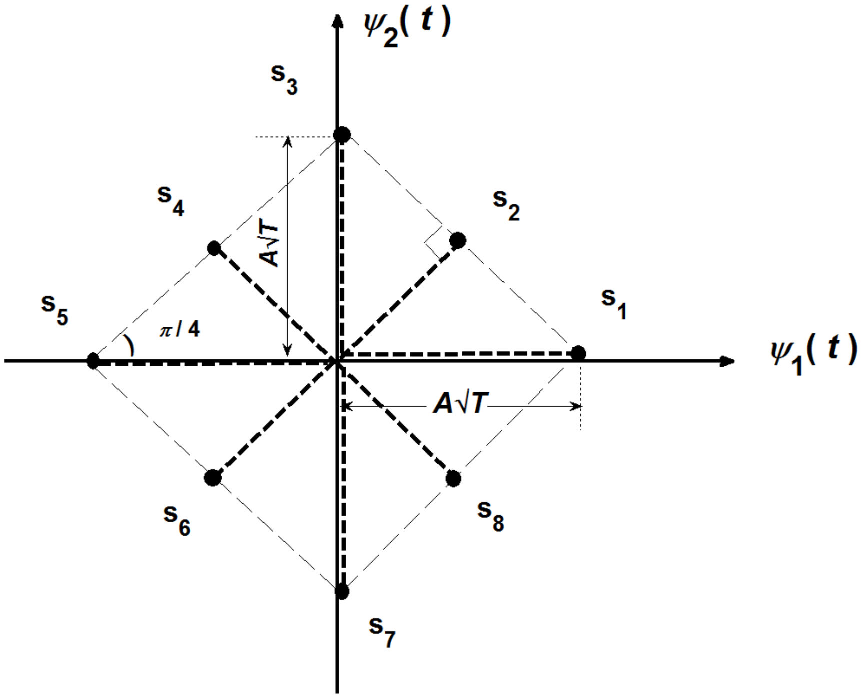

Fig. 1.1 Constellation for Q1.

Solution : a. Looking at the constellation in Fig. 1.1, we see that the signal vector lengths, thus the related energies are unequal and the signal space in this constellation contains two dimensions, thus constellation is QAM.

By adapting the following orthonormalized basis functions,

$$\psi_1(t)=\begin{cases}\sqrt{2/T} & 0\le t\le T/2\\ 0 & \text{otherwise}\end{cases}\qquad \psi_2(t)=\begin{cases}\sqrt{2/T} & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases} \tag{1.1}$$

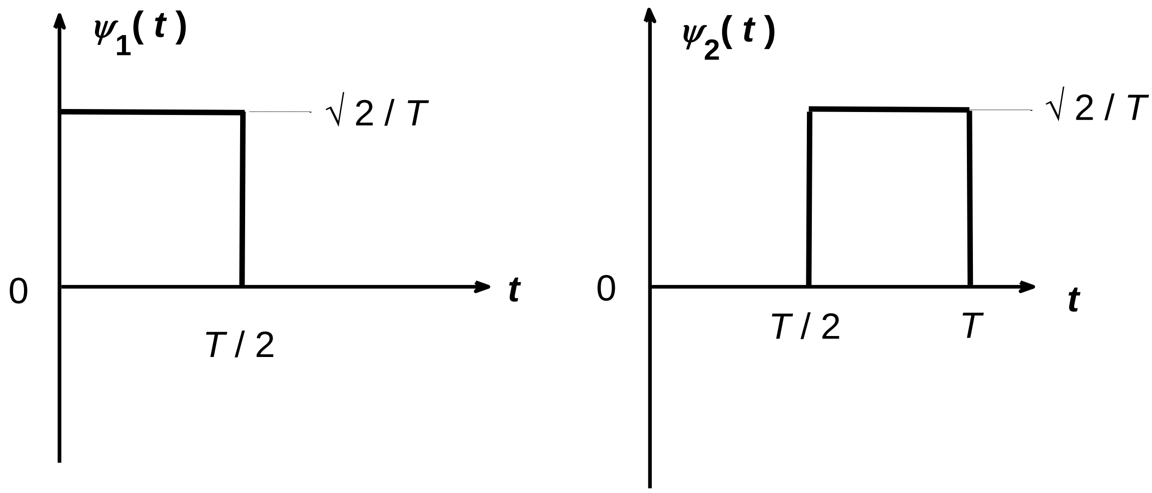

Fig. 1.2 The orthonormalized basis functions for Q1.

By using (1), Figs. 1.1 and 1.2 we obtain the followings for the time waveforms of $s_1(t)\cdots s_8(t)$

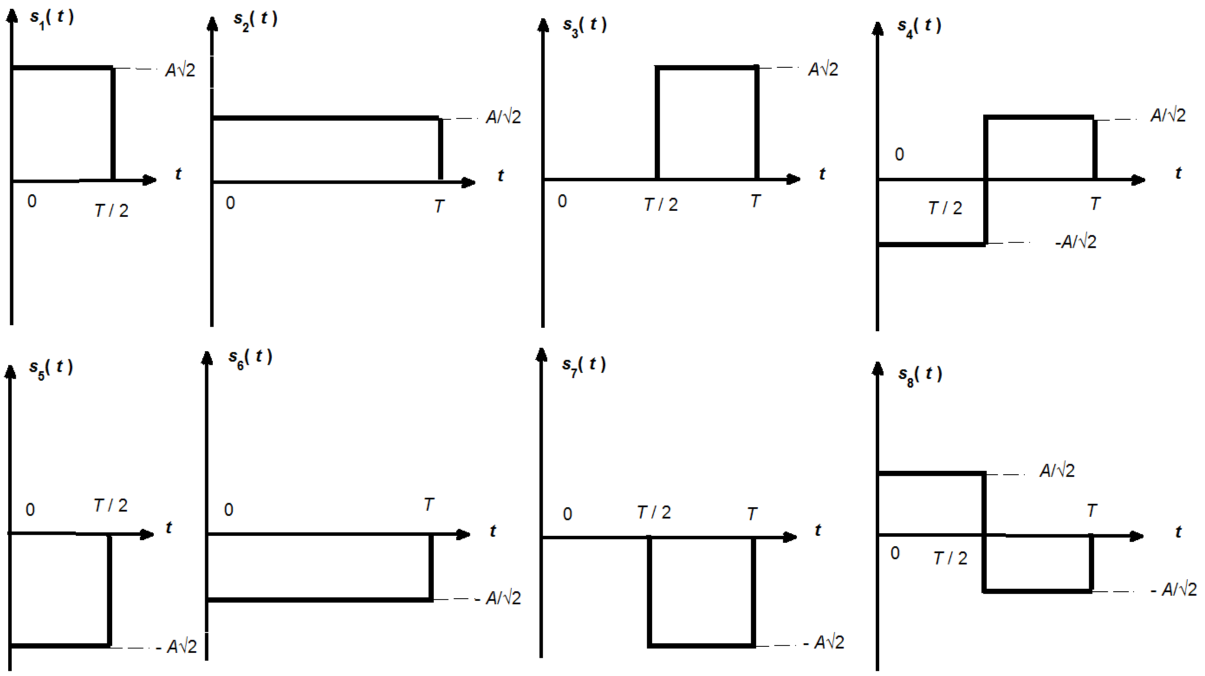

Fig. 1.3 Time waveforms of $s_1(t)\cdots s_8(t)$ for Q1.

The related time waveform expressions for $s_1(t)\cdots s_8(t)$ are given in (1.2).

$$s_1(t)=\begin{cases} A\sqrt{2} & 0\le t\le T/2 \\ 0 & \text{otherwise}\end{cases},\ s_1(t)=A\sqrt{T}\psi_1(t),\ \mathbf{s_1}=[s_{11},\ s_{12}]=\left[A\sqrt{T},\ 0\right],\ \varepsilon_{s_1^a}=\|\mathbf{s_1}\|^2=A^2T$$

$$s_2(t)=\begin{cases} A/\sqrt{2} & 0\le t\le T \\ 0 & \text{otherwise}\end{cases},\ s_2(t)=\frac{A}{2}\sqrt{T}\psi_1(t)+\frac{A}{2}\sqrt{T}\psi_2(t),\ \mathbf{s_2}=[s_{21},\ s_{22}]=\left[\frac{A}{2}\sqrt{T},\ \frac{A}{2}\sqrt{T}\right],\ \varepsilon_{s_2^a}=\|\mathbf{s_2^a}\|^2=\frac{A^2T}{2}$$

$$s_3(t)=\begin{cases} A\sqrt{2} & T/2\le t\le T \\ 0 & \text{otherwise}\end{cases},\ s_3(t)=A\sqrt{T}\psi_2(t),\ \mathbf{s_3}=[s_{31},\ s_{32}]=\left[0,\ A\sqrt{T}\right],\ \varepsilon_{s_3^a}=A^2T$$

$$s_4(t)=\begin{cases} -A/\sqrt{2} & 0\le t\le T/2 \\ A/\sqrt{2} & T/2\le t\le T \\ 0 & \text{otherwise}\end{cases},\ s_4(t)=-\frac{A}{2}\sqrt{T}\psi_1(t)+\frac{A}{2}\sqrt{T}\psi_2(t),\ \mathbf{s_4}=[s_{41},\ s_{42}]=\left[-\frac{A}{2}\sqrt{T},\ \frac{A}{2}\sqrt{T}\right],\ \varepsilon_{s_4^a}=\frac{A^2T}{2}$$

$$s_5(t)=\begin{cases} -A\sqrt{2} & 0\le t\le T/2 \\ 0 & \text{otherwise}\end{cases},\ s_5(t)=-A\sqrt{T}\psi_1(t),\ \mathbf{s_5}=[s_{51},\ s_{52}]=\left[-A\sqrt{T},\ 0\right],\ \varepsilon_{s_5^a}=A^2T$$

$$s_6(t)=\begin{cases} -A/\sqrt{2} & 0\le t\le T \\ 0 & \text{otherwise}\end{cases},\ s_6(t)=-\frac{A}{2}\sqrt{T}\psi_1(t)-\frac{A}{2}\sqrt{T}\psi_2(t),\ \mathbf{s_6}=[s_{61},\ s_{62}]=\left[-\frac{A}{2}\sqrt{T},\ -\frac{A}{2}\sqrt{T}\right],\ \varepsilon_{s_6^a}=\frac{A^2T}{2}$$

$$s_7(t)=\begin{cases} -A\sqrt{2} & T/2\le t\le T \\ 0 & \text{otherwise}\end{cases},\ s_7(t)=-A\sqrt{T}\psi_2(t),\ \mathbf{s_7}=[s_{71},\ s_{72}]=\left[0,\ -A\sqrt{T}\right],\ \varepsilon_{s_7^a}=A^2T$$

$$s_8(t)=\begin{cases} A\sqrt{2} & 0\le t\le T/2 \\ -A/\sqrt{2} & T/2\le t\le T \\ 0 & \text{otherwise}\end{cases},\ s_8(t)=\frac{A}{2}\sqrt{T}\psi_1(t)-\frac{A}{2}\sqrt{T}\psi_2(t),\ \mathbf{s_8}=[s_{81},\ s_{82}]=\left[\frac{A}{2}\sqrt{T},\ -\frac{A}{2}\sqrt{T}\right],\ \varepsilon_{s_8^a}=\frac{A^2T}{2} \quad (1.2)$$

The total energy used if all signals are transmitted with equal probability, $E_T=4\times A^2T+4\times\frac{A^2T}{2}=6A^2T$

The distances between signal vector ends are given in (1.3)

$$\begin{aligned} d_{12}&=d_{23}=d_{34}=d_{45}=d_{56}=d_{67}=A\sqrt{T/2}\\ d_{13}&=d_{35}=d_{57}=d_{71}=A\sqrt{2T}\\ d_{24}&=d_{46}=d_{68}=d_{82}=d_{15}=d_{37}=A\sqrt{T}\\ d_{26}&=d_{48}=A\sqrt{2T}\\ d_{14}&=d_{16}=d_{38}=A\sqrt{5T/2} \end{aligned} \quad (1.3)$$

b. Looking at constellation A, we see that there are two types of decision boundaries. They can be described by examining the cases of $\mathbf{s_1}$ and $\mathbf{s_2}$. The received vector, $\mathbf{r}$ that we supply to the detector for the two cases will become

$$\mathbf{r_1}=\begin{bmatrix} r_{11}\\ r_{12}\end{bmatrix}=\begin{bmatrix} A\sqrt{T}+n_1\\ n_2\end{bmatrix} \quad , \quad \mathbf{r_2}=\begin{bmatrix} r_{21}\\ r_{22}\end{bmatrix}=\begin{bmatrix} \frac{A}{2}\sqrt{T}+n_1\\ \frac{A}{2}\sqrt{T}+n_2\end{bmatrix} \quad (1.4)$$

Based on the observations in ECE 588MT-19112012_Solutions, it is sufficient to evaluate correlation metrics $C_1(\mathbf{r_1},\mathbf{s_m})$ at $m=1,\ 2,\ 8,\ 5$ for $\mathbf{s_1}$, $C_2(\mathbf{r_2},\mathbf{s_m})$ at $m=1,\ 2,\ 3,\ 4,\ 6,\ 8$ and for $\mathbf{s_2}$ (note that additionally we have chosen 4 and 8), the others will be covered in these analyses.

$$m=1\ ,\ C_1(\mathbf{r_1},\mathbf{s_1})=2\,\mathbf{s_1}\cdot\mathbf{r_1}-\|\mathbf{s_1}\|^2=2\left[A\sqrt{T},\ 0\right]\begin{bmatrix}A\sqrt{T}+n_1\\ n_2\end{bmatrix}-A^2T=A^2T+2An_1\sqrt{T}$$

$$m=2\ ,\ C_1(\mathbf{r_1},\mathbf{s_2})=2\,\mathbf{s_2}\cdot\mathbf{r_1}-\|\mathbf{s_2}\|^2=2\left[\frac{A}{2}\sqrt{T},\ A\sqrt{T}\right]\begin{bmatrix}A\sqrt{T}+n_1\\ n_2\end{bmatrix}-\frac{A^2T}{2}=\frac{A^2T}{2}+An_1\sqrt{T}+An_2\sqrt{T}$$

$$m=5\ ,\ C_1(\mathbf{r_1},\mathbf{s_5})=2\,\mathbf{s_5}\cdot\mathbf{r_1}-\|\mathbf{s_5}\|^2=2\left[-A\sqrt{T},\ 0\right]\begin{bmatrix}A\sqrt{T}+n_1\\ n_2\end{bmatrix}-A^2T=-3A^2T-2An_1\sqrt{T}$$

$$m=8\ ,\ C_1(\mathbf{r_1},\mathbf{s_8})=2\,\mathbf{s_8}\cdot\mathbf{r_1}-\|\mathbf{s_8}\|^2=2\left[\frac{A}{2}\sqrt{T},\ -\frac{A}{2}\sqrt{T}\right]\begin{bmatrix}A\sqrt{T}+n_1\\ n_2\end{bmatrix}-\frac{A^2T}{2}=\frac{A^2T}{2}+An_1\sqrt{T}-An_2\sqrt{T}$$

$$m=1\ ,\ C_2(\mathbf{r_2},\mathbf{s_1})=2\,\mathbf{s_1}\cdot\mathbf{r_2}-\|\mathbf{s_1}\|^2=2\left[A\sqrt{T},\ 0\right]\begin{bmatrix}\frac{A}{2}\sqrt{T}+n_1\\ \frac{A}{2}\sqrt{T}+n_2\end{bmatrix}-A^2T=2An_1\sqrt{T}$$

$$m=2\ ,\ C_2(\mathbf{r_2},\mathbf{s_2})=2\,\mathbf{s_2}\cdot\mathbf{r_2}-\|\mathbf{s_2}\|^2=2\left[\frac{A}{2}\sqrt{T},\ \frac{A}{2}\sqrt{T}\right]\begin{bmatrix}\frac{A}{2}\sqrt{T}+n_1\\ \frac{A}{2}\sqrt{T}+n_2\end{bmatrix}-\frac{A^2T}{2}=\frac{A^2T}{2}+An_1\sqrt{T}+An_2\sqrt{T}$$

$$m=3\ ,\ C_2(\mathbf{r_2},\mathbf{s_3})=2\,\mathbf{s_3}\cdot\mathbf{r_2}-\|\mathbf{s_3}\|^2=2\left[0,\ A\sqrt{T}\right]\begin{bmatrix}\frac{A}{2}\sqrt{T}+n_1\\ \frac{A}{2}\sqrt{T}+n_2\end{bmatrix}-A^2T=2An_2\sqrt{T}$$

$$m=4\ ,\ C_2(\mathbf{r_2},\mathbf{s_4})=2\,\mathbf{s_4}\cdot\mathbf{r_2}-\|\mathbf{s_4}\|^2=2\left[-\frac{A}{2}\sqrt{T},\ \frac{A}{2}\sqrt{T}\right]\begin{bmatrix}\frac{A}{2}\sqrt{T}+n_1\\ \frac{A}{2}\sqrt{T}+n_2\end{bmatrix}-\frac{A^2T}{2}=-\frac{A^2T}{2}-An_1\sqrt{T}+An_2\sqrt{T}$$

$$m=6\ ,\ C_2(\mathbf{r_2},\mathbf{s_6})=2\,\mathbf{s_6}\cdot\mathbf{r_2}-\|\mathbf{s_6}\|^2=2\left[-\frac{A}{2}\sqrt{T},\ -\frac{A}{2}\sqrt{T}\right]\begin{bmatrix}\frac{A}{2}\sqrt{T}+n_1\\ \frac{A}{2}\sqrt{T}+n_2\end{bmatrix}-\frac{A^2T}{2}=-\frac{3A^2T}{2}-An_1\sqrt{T}-An_2\sqrt{T}$$

$$m=8\ ,\ C_2(\mathbf{r_2},\mathbf{s_8})=2\,\mathbf{s_8}\cdot\mathbf{r_2}-\|\mathbf{s_8}\|^2=2\left[\frac{A}{2}\sqrt{T},\ -\frac{A}{2}\sqrt{T}\right]\begin{bmatrix}\frac{A}{2}\sqrt{T}+n_1\\ \frac{A}{2}\sqrt{T}+n_2\end{bmatrix}-\frac{A^2T}{2}=-\frac{A^2T}{2}+An_1\sqrt{T}-An_2\sqrt{T} \qquad (1.5)$$

To get the correct decision regions we must have

$$\begin{aligned}&\text{For } \mathbf{s_1}: C_1(\mathbf{r_1},\mathbf{s_1})>C_1(\mathbf{r_1},\mathbf{s_2}),\ C_1(\mathbf{r_1},\mathbf{s_1})>C_1(\mathbf{r_1},\mathbf{s_5}),\ C_1(\mathbf{r_1},\mathbf{s_1})>C_1(\mathbf{r_1},\mathbf{s_8})\\ &\text{For } \mathbf{s_2}: C_2(\mathbf{r_2},\mathbf{s_2})>C_2(\mathbf{r_1},\mathbf{s_1}),\ C_2(\mathbf{r_2},\mathbf{s_2})>C_2(\mathbf{r_2},\mathbf{s_3}),\ C_2(\mathbf{r_2},\mathbf{s_2})>C_2(\mathbf{r_2},\mathbf{s_4})\\ &\qquad C_2(\mathbf{r_2},\mathbf{s_2})>C_2(\mathbf{r_2},\mathbf{s_6}),\ C_2(\mathbf{r_2},\mathbf{s_2})>C_2(\mathbf{r_2},\mathbf{s_8})\end{aligned} \qquad (1.6)$$

Hence, we get the following inequalities

$$C_1(\mathbf{r}_1,\mathbf{s}_1) > C_1(\mathbf{r}_1,\mathbf{s}_2) : A^2T + 2An_1\sqrt{T} > \frac{A^2T}{2} + An_1\sqrt{T} + An_2\sqrt{T} \rightarrow \overbrace{A\sqrt{T} + n_1}^{r_{11}} + \overset{r_{11}-A\sqrt{T}}{n_1} > 2\overset{r_{12}}{n_2}$$

$$\rightarrow r_{11} - 0.5A\sqrt{T} > r_{12}$$

$$C_1(\mathbf{r}_1,\mathbf{s}_1) > C_1(\mathbf{r}_1,\mathbf{s}_5) : A^2T + 2An_1\sqrt{T} > -3A^2T - 2An_1\sqrt{T} \rightarrow r_{11} > 0$$

$$C_1(\mathbf{r}_1,\mathbf{s}_1) > C_1(\mathbf{r}_1,\mathbf{s}_8) : A^2T + 2An_1\sqrt{T} > \frac{A^2T}{2} + An_1\sqrt{T} - An_2\sqrt{T} \rightarrow -r_{11} + 0.5A\sqrt{T} < r_{12}$$

$$C_2(\mathbf{r}_2,\mathbf{s}_2) > C_2(\mathbf{r}_2,\mathbf{s}_1) : \frac{A^2T}{2} + An_1\sqrt{T} + An_2\sqrt{T} > 2An_1\sqrt{T} \rightarrow r_{21} - 0.5A\sqrt{T} < r_{22}$$

$$C_2(\mathbf{r}_2,\mathbf{s}_2) > C_2(\mathbf{r}_2,\mathbf{s}_3) : \frac{A^2T}{2} + An_1\sqrt{T} + An_2\sqrt{T} > 2An_2\sqrt{T} \rightarrow r_{21} + 0.5A\sqrt{T} > r_{22}$$

$$C_2(\mathbf{r}_2,\mathbf{s}_2) > C_2(\mathbf{r}_2,\mathbf{s}_4) : \frac{A^2T}{2} + An_1\sqrt{T} + An_2\sqrt{T} > -\frac{A^2T}{2} - An_1\sqrt{T} + An_2\sqrt{T} \rightarrow r_{21} > 0$$

$$C_2(\mathbf{r}_2,\mathbf{s}_2) > C_2(\mathbf{r}_2,\mathbf{s}_6) : \frac{A^2T}{2} + An_1\sqrt{T} + An_2\sqrt{T} > -\frac{3A^2T}{2} - An_1\sqrt{T} - An_2\sqrt{T} \rightarrow -r_{21} < r_{22}$$

$$C_2(\mathbf{r}_2,\mathbf{s}_2) > C_2(\mathbf{r}_2,\mathbf{s}_8) : \frac{A^2T}{2} + An_1\sqrt{T} + An_2\sqrt{T} > -\frac{A^2T}{2} + An_1\sqrt{T} - An_2\sqrt{T} \rightarrow r_{22} > 0 \qquad (1.7)$$

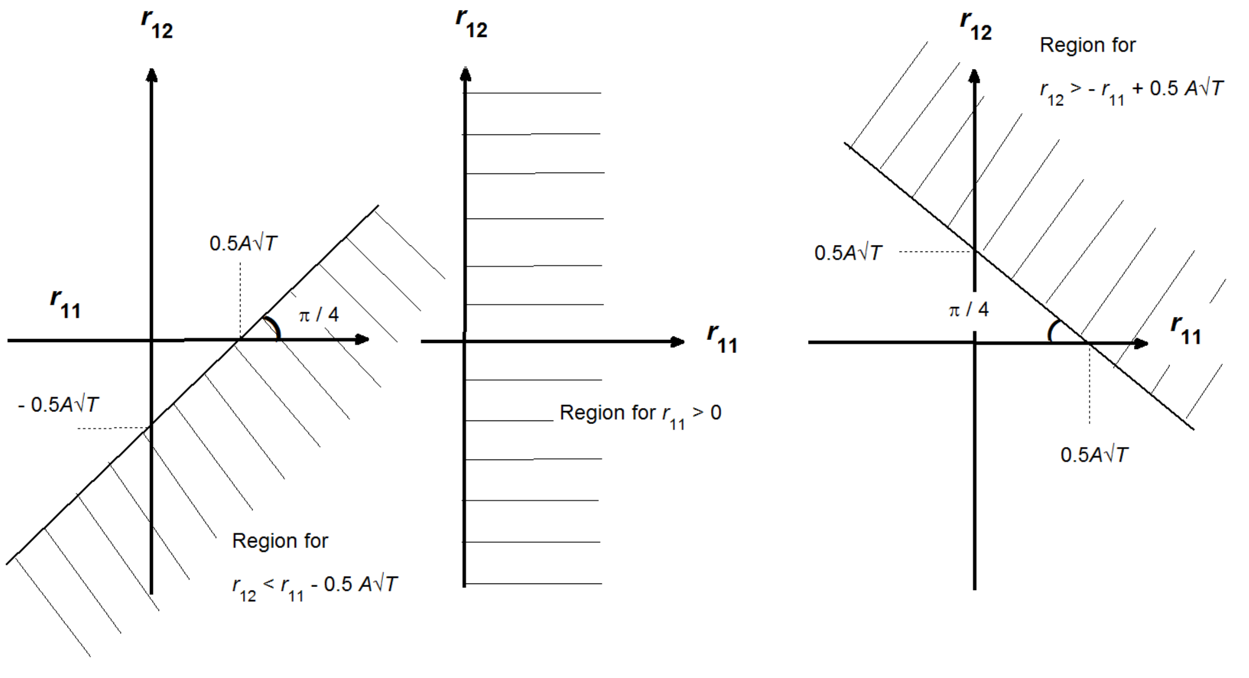

Fig. 1. 4 Regions for conditions in (1.7) for $\mathbf{s}_1$.

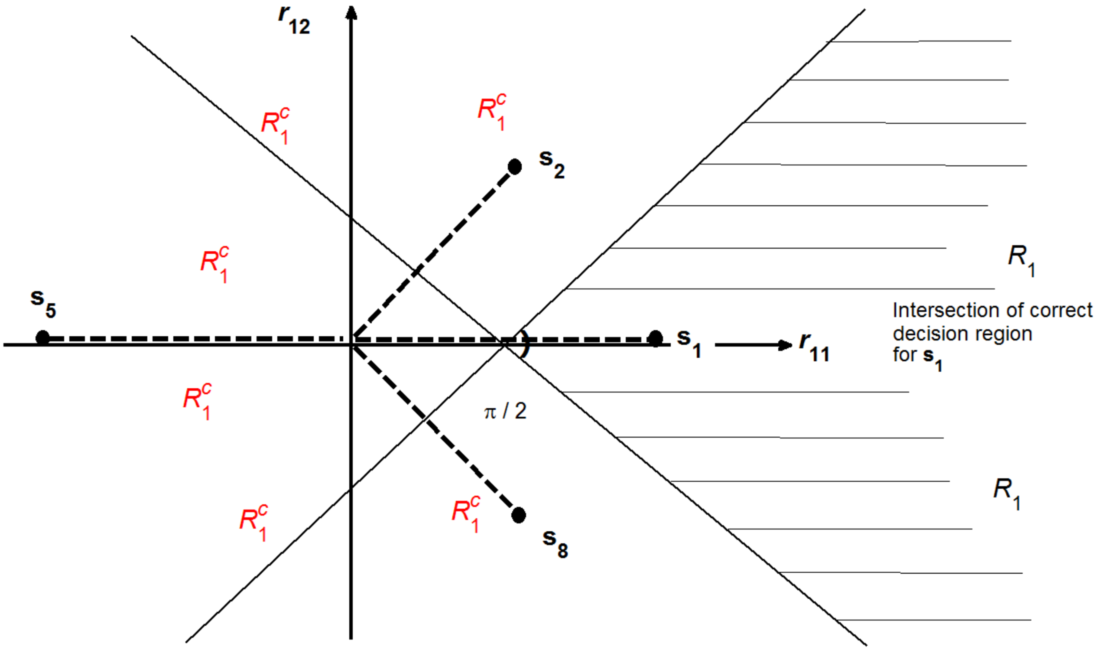

Fig. 1. 5 Intersection of correct decision region for conditions in (1.7). for $\mathbf{s}_1$.

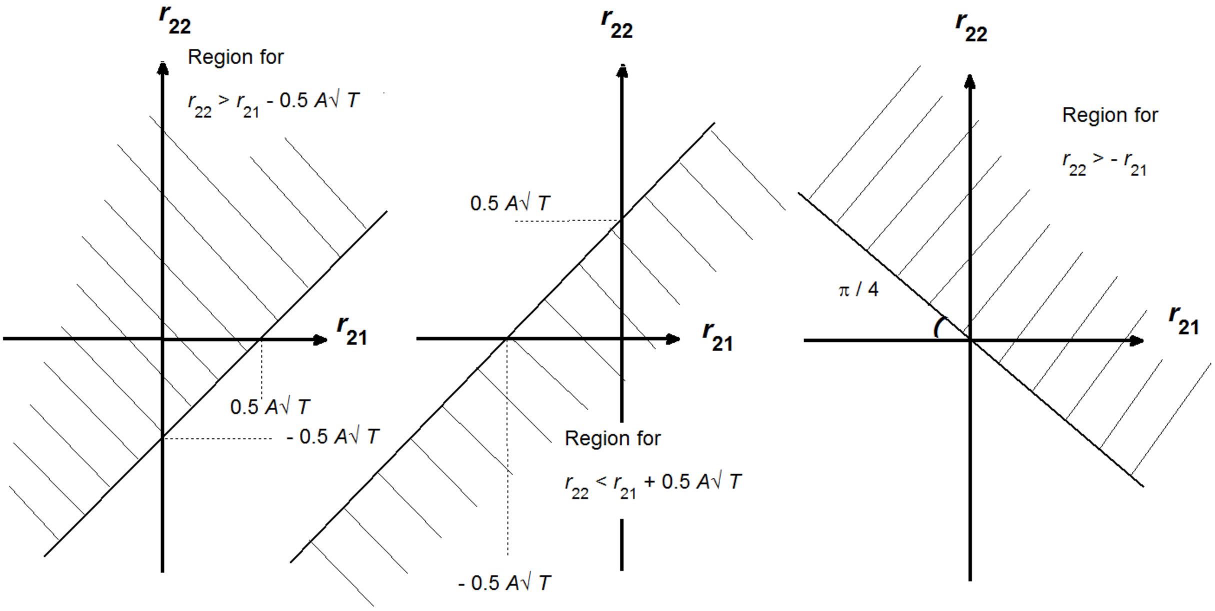

Fig. 1. 6 Regions for conditions in (1.7) for $\mathbf{s}_2$. Note that the simple cases of $C_2(\mathbf{r}_2, \mathbf{s}_2) > C_2(\mathbf{r}_2, \mathbf{s}_4)$ and $C_2(\mathbf{r}_2, \mathbf{s}_2) > C_2(\mathbf{r}_2, \mathbf{s}_8)$ are not illustrated.

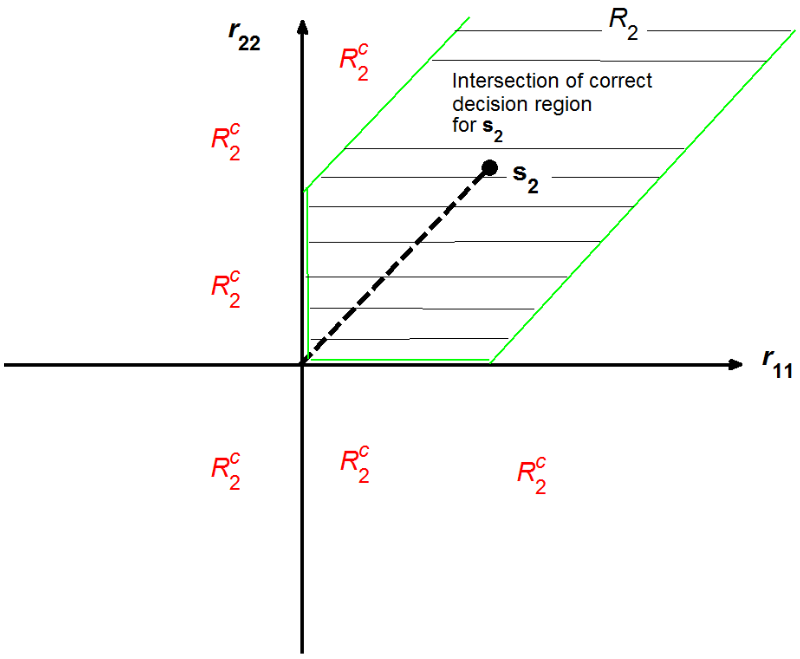

Fig. 1. 7 Intersection of correct decision region for conditions in (1.7). for $\mathbf{s}_2$.

As seen from above, signal vectors, $\mathbf{s}_1$, $\mathbf{s}_3$, $\mathbf{s}_5$ and $\mathbf{s}_7$ will have identical correct decision regions, thus the probability of error for $\mathbf{s}_1$, $\mathbf{s}_3$, $\mathbf{s}_5$ and $\mathbf{s}_7$ will be the same. On the other hand, signal vectors, $\mathbf{s}_2$, $\mathbf{s}_4$, $\mathbf{s}_6$ and $\mathbf{s}_8$ will have identical correct decision regions, thus the probability of error for $\mathbf{s}_2$, $\mathbf{s}_4$, $\mathbf{s}_6$ and $\mathbf{s}_8$ will be the same.

2. (30 Points) Answer the following questions as **True** or **False.** For the **False** ones give the correct answer or the reason. For the **True** ones, justify your answer

a) PSK has more than 2 signal vectors : True, but this can be the case for any modulation type, i.e. not specific to PSK.

b) The probability of error in ASK is expected to be lower than the one in QAM : False, as seen from the Figs. 6.13 and 6.14 of lecture notes entitled, " ECE376_ Dimensionality of Signals_ASK_PSK_QAM_FSK_Jan 2013_HTE".

c) QAM is a combination of ASK and FSK : False, since QAM is a combination of ASK and PSK.

d) In generation of FM, we can use reactive circuit elements : True, as explained on pp. 12 of lecture notes entitled, " ECE 376_AM_FM Demodulation_Jan 2013_HTE".

e) The noise performance of FM is better than PM : True, since as there is bandwidth expansion during modulation and bandwidth contraction during demodulation in FM, whereas this does not take place in PM.

f) In demodulation of FM, it is essential to use phase synchronized local carrier : True as illustrated in Fig. 2.4 of lecture notes entitled, " ECE 376_AM_FM Demodulation_Jan 2013_HTE".