Çankaya University – ECE Department – ECE 376

Student Name :
Student Number :

Duration : 2 hours
Open book exam

Questions

1. (70 Points) The following two set of waveforms $s_1(t)$, $s_2(t)$, $s_3(t)$ and $s_1'(t)$, $s_2'(t)$, $s_3'(t)$ are given. For each set, find and plot either by eye inspection or by Gram-Schmidt orthonormalization procedure, the orthonormal basis functions $\psi_1(t) \cdots \psi_N(t)$ and $\psi_1'(t) \cdots \psi_{N'}'(t)$, thus determine the dimensionality of each set. What type of modulations $s_1(t)$, $s_2(t)$, $s_3(t)$ and $s_1'(t)$, $s_2'(t)$, $s_3'(t)$ represent ?. Make comments about the bandwidth requirement of each signal set. Find the expressions of $s_1(t)$, $s_2(t)$, $s_3(t)$ and $s_1'(t)$, $s_2'(t)$, $s_3'(t)$ in terms of its own orthonormal basis functions. Plot the constellation diagram for each signal set, also indicating the distances between signal vector ends. Make comments about minimum distance of each constellation. For a receiver made up of correlator and matched filter demodulators, draw the block diagrams relevant to the given signal sets . Find the outputs of correlators and matched filters if $s_1(t)$ of the first signal set and $s_1'(t)$ of the other signal set were transmitted from transmitter and get mixed up with AWGN at receiver.

$$s_1(t) = A \qquad 0 \le t \le T$$

$$s_2(t) = \begin{cases} A/2 & 0 \le t \le T/2 \\ -A/2 & T/2 \le t \le T \end{cases}$$

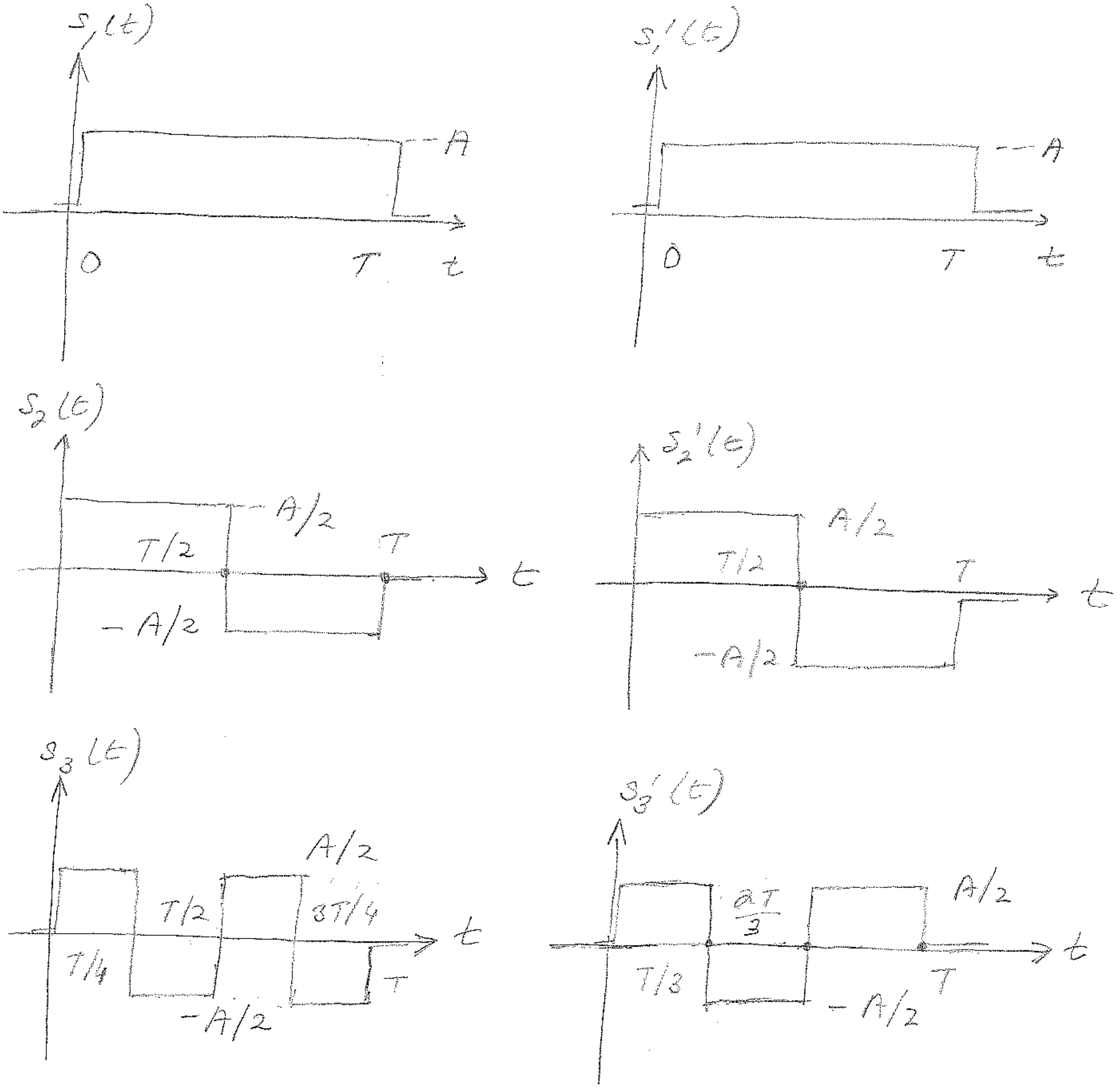
$$s_3(t) = \begin{cases} A/2 & 0 \le t \le T/4 \\ -A/2 & T/4 \le t \le T/2 \\ A/2 & T/2 \le t \le 3T/4 \\ -A/2 & 3T/4 \le t \le T \end{cases}$$

$$s_1'(t) = A \qquad 0 \le t \le T$$

$$s_2'(t) = \begin{cases} A/2 & 0 \le t \le T/2 \\ -A/2 & T/2 \le t \le T \end{cases}$$

$$s_3'(t) = \begin{cases} A/2 & 0 \le t \le T/3 \\ -A/2 & T/3 \le t \le 2T/3 \\ A/2 & 2T/3 \le t \le T \end{cases}$$

Solution : From the given expressions, we plot $s_1(t)$ -- $s_3(t)$ and $s_1'(t)$ --- $s_3'(t)$ as shown overleaf



Note that $s_1(t)$ --- $s_3(t)$ are orthogonal set (but not normalized), but $s_1'(t)$ -- $s_3'(t)$ do not satisfy the orthogonality condition as a whole set.

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Thus, for $s_1(t) \cdots s_3(t)$, orthonormal basis functions are simply the same same but with energies normalized to unity.

$$\mathcal{E}_{s_1} = A^2 T, \quad \mathcal{E}_{s_2} = \mathcal{E}_{s_3} = A^2 T/4$$

Note that $s_1(t)$ contains DC + AC energy (due to DC shift of $A/2$)

$$\psi_1(t) = \frac{s_1(t)}{\mathcal{E}_{s_1}^{1/2}} = \frac{1}{\sqrt{T}} \quad 0 < t < T, \; 0 \text{ elsewhere}$$

$$\psi_2(t) = \frac{s_2(t)}{\mathcal{E}_{s_2}^{1/2}} = \begin{cases} \frac{1}{\sqrt{T}} & 0 < t < T/2 \\ -\frac{1}{\sqrt{T}} & T/2 < t < T \end{cases}$$

$$\psi_3(t) = \frac{s_3(t)}{\mathcal{E}_{s_3}^{1/2}} = \begin{cases} \frac{1}{\sqrt{T}} & 0 < t < T/4 \\ -\frac{1}{\sqrt{T}} & T/4 < t < T/2 \\ \frac{1}{\sqrt{T}} & \frac{T}{2} < t < \frac{3T}{4} \\ -1/\sqrt{T} & \frac{3T}{4} < t < T \end{cases}$$

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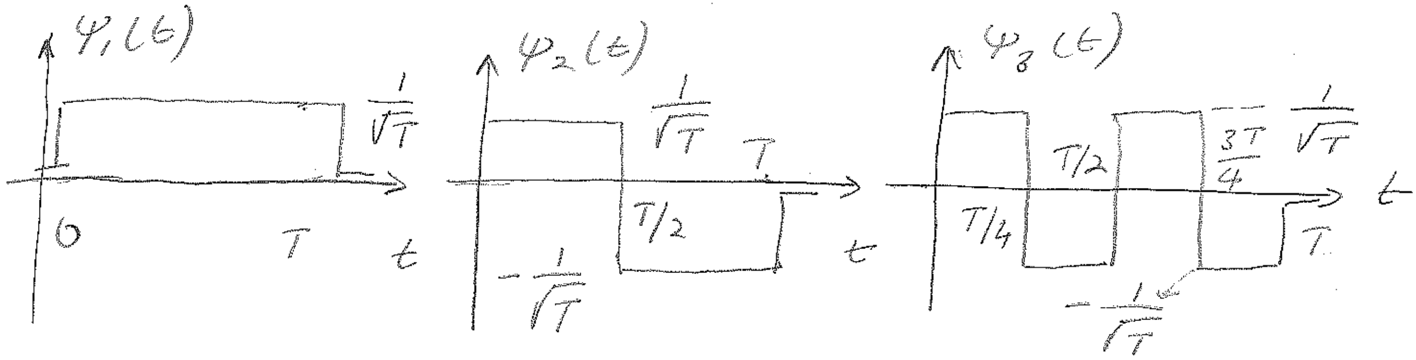
$\varphi_1(t)$ --- $\varphi_3(t)$ are shown below



Expressing $s_1(t)$ -- $s_3(t)$ in terms of $\varphi_1(t)$ -- $\varphi_3(t)$

and writing for signal vectors s_1 --- s_3, we get

$s_1(t) = AT^{1/2}\varphi_1(t) = A \quad 0<t<T, \quad s_1 = [AT^{1/2} \; 0 \; 0]$

$$s_2(t) = \frac{AT^{1/2}}{2}\varphi_2(t) = \begin{cases} A/2 & 0<t<T/2 \\ -A/2 & T/2<t<T \end{cases}$$

$$s_2 = \left[0 \quad \frac{AT^{1/2}}{2} \quad 0\right]$$

$$s_3(t) = \frac{AT^{1/2}}{2}\varphi_3(t) = \begin{cases} A/2 & 0<t<T/4 \\ -A/2 & T/4<t<T/2 \\ A/2 & T/2<t<3T/4 \\ -A/2 & 3T/4<t<T \end{cases}$$

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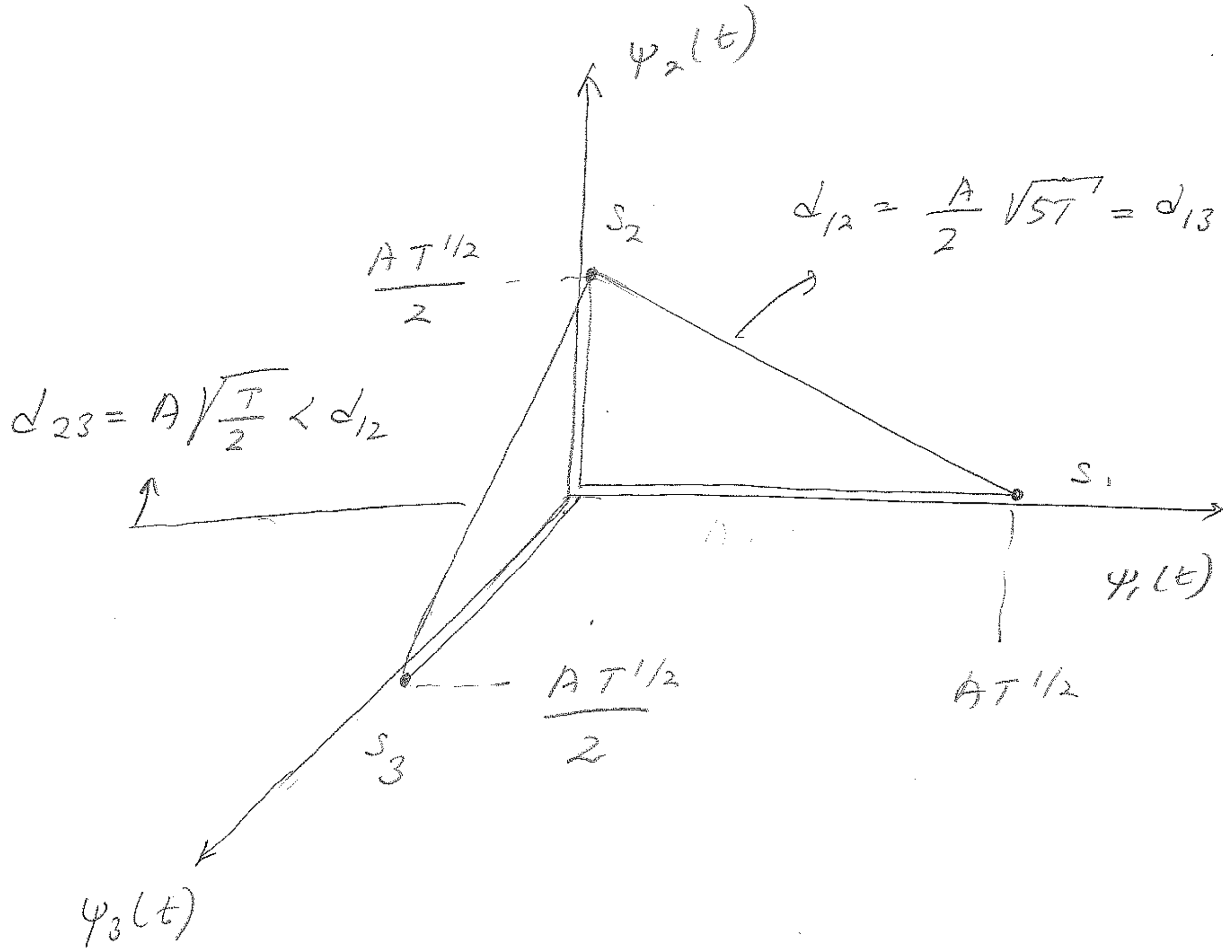
$s_3 = \left[0 \quad 0 \quad \frac{AT^{1/2}}{2} \right]$

So the constellation diagram for $s_1(t) -- s_3(t)$ looks like the following



Note that for $s_1(t) --- s_3(t)$, $M = 3$, $N = 3$ thus this is multidimensional $(N > 2)$ modulation

Now coming to $s_1'(t)$ --- $s_3'(t)$, we see that

$s_1'(t) = s_1(t)$, $s_2'(t) = s_2(t)$ but $s_3'(t) \neq s_3(t)$

This means $\varphi_1'(t) = \varphi_1(t)$, $\varphi_2'(t) = \varphi_2(t)$ but

$\varphi_3'(t) \neq \varphi_3(t)$. To find $\varphi_3'(t)$ we formally

use the Gram-Schmidt Orthonormalization procedure

explained on pp. 341 - 342 in Proakis 2002 book.

1) $\varphi_3'(t) = \dfrac{d_3(t)}{\mathcal{E}_{d_3}^{1/2}}$, $d_3(t) = s_3(t) - \sum_{i=1}^{k-1} c_{ki}\varphi_i(t)$

$c_{ki} = \int_0^T s_k(t)\,\varphi_i(t)\,dt$

2) In our case $k=3$, $i = 1 \cdots k-1$, so

there is only two terms of c_{ki}, namely

c_{31} and c_{32}

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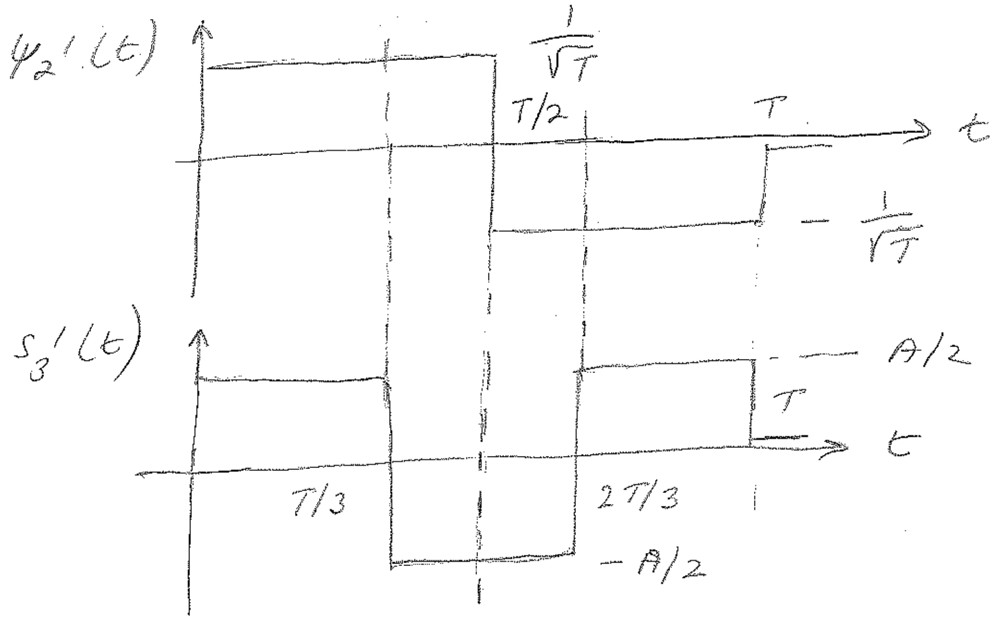
3) $c_{31} = \int_0^T s_3'(t)\,\varphi_1'(t)\,dt$, By taking into account the different time slices of $s_3(t)$

$$= \frac{A}{2\sqrt{T}} \int_0^{T/3} dt - \frac{A}{2\sqrt{T}} \int_{T/3}^{2T/3} dt$$

$$+ \frac{A}{2\sqrt{T}} \int_{2T/3}^{T} dt = \frac{A}{6} T^{1/2}$$

$$c_{32} = \int_0^T s_3'(t)\,\varphi_2'(t)\,dt = \frac{A}{2\sqrt{T}} \int_0^{T/2} dt - \frac{A}{2\sqrt{T}} \int_{T/3}^{T/2} dt$$

$$+ \frac{A}{2\sqrt{T}} \int_{T/2}^{2T/3} dt - \frac{A}{2\sqrt{T}} \int_{2T/3}^{T} dt = 0$$, illustrated below



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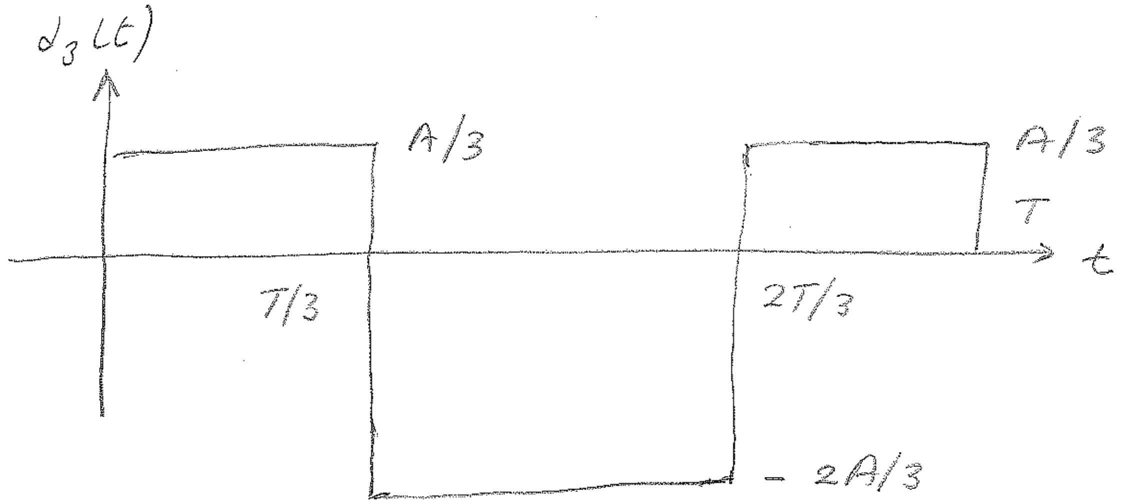
4) So $d_3(t) = s_3'(t) - c_{31}\varphi_1(t)$

$$d_3(t) = \begin{cases} A/3 & 0 < t < T/3 \\ -2A/3 & T/3 < t < 2T/3 \\ A/3 & 2T/3 < t < T \end{cases}$$



$$\mathcal{E}_{d_3} = \frac{A^2}{9} \times \frac{T}{3} + \frac{4A^2}{9} \times \frac{T}{3} + \frac{A^2}{9} \times \frac{T}{3} = \frac{2A^2T}{9}$$

$\mathcal{E}_{d_3}^{1/2} = \frac{A}{3}\sqrt{2T}$, finally $\varphi_3'(t)$ will be

$$\varphi_3'(t) = \begin{cases} \frac{1}{\sqrt{2T}} & 0 < t < T/3 \\ -\sqrt{\frac{2}{T}} & T/2 < t < 2T/3 \\ \frac{1}{\sqrt{2T}} & \frac{2T}{3} < t < T \end{cases}$$

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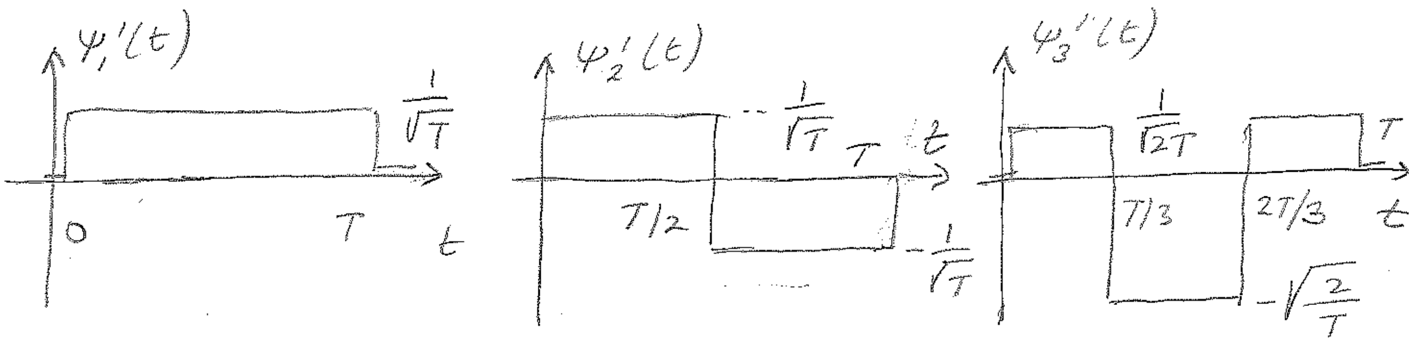
It is easy to verify that

a) $\int_0^T [\varphi_3'(t)]^2 dt = 1$; Energy is normalized

b) $\int_0^T \varphi_3'(t)\,\varphi_1'(t)\,dt = \int_0^T \varphi_3'(t)\,\varphi_2'(t)\,dt = 0$

$\varphi_1'(t)$ -- $\varphi_3'(t)$ are plotted below



So $s_1'(t)$ -- $s_3'(t)$ are also multidimensional

with $M = N = 3$.

Expressing $s_1'(t)$ -- $s_3'(t)$ in terms of $\varphi_1'(t)$ -- $\varphi_3'(t)$

we find

$s_1'(t) = AT^{1/2}\varphi_1'(t)$, $s_1' = [AT^{1/2} \;\; 0 \;\; 0]$

$$s_2'(t) = \frac{A\,T^{1/2}}{2}\,\varphi_2'(t)\ ,\quad s_2' = \left[0 \quad \frac{AT^{1/2}}{2} \quad 0\right]$$

To determine how to express $s_3'(t)$ in terms of $\varphi_1'(t)$, $\varphi_2'(t)$ and $\varphi_3'(t)$, we resort to the formal method of Gram-Schmidt again, thus

$$s_3'(t) = \sum_{n=1}^{N} s_{3n}'\,\varphi_n'(t),\quad s_{3n}' = \int_0^T s_3'(t)\,\varphi_n'(t)\,dt$$

so $$S_{31} = \int_0^T s_3'(t)\,\varphi_1'(t)\,dt = \frac{A}{2}\frac{1}{\sqrt{T}}\int_0^{T/3} dt$$

$$-\frac{A}{2}\frac{1}{\sqrt{T}}\int_{T/3}^{2T/3} dt + \frac{A}{2}\frac{1}{\sqrt{T}}\int_{2T/3}^{T} dt = \frac{AT^{1/2}}{6}$$

$$S_{32} = \int_0^T s_3'(t)\,\varphi_2'(t)\,dt = 0$$

$$S_{33} = \int_0^T s_3'(t)\,\varphi_3'(t)\,dt = \frac{A}{2}\frac{1}{\sqrt{2T}}\int_0^{T/3} dt$$

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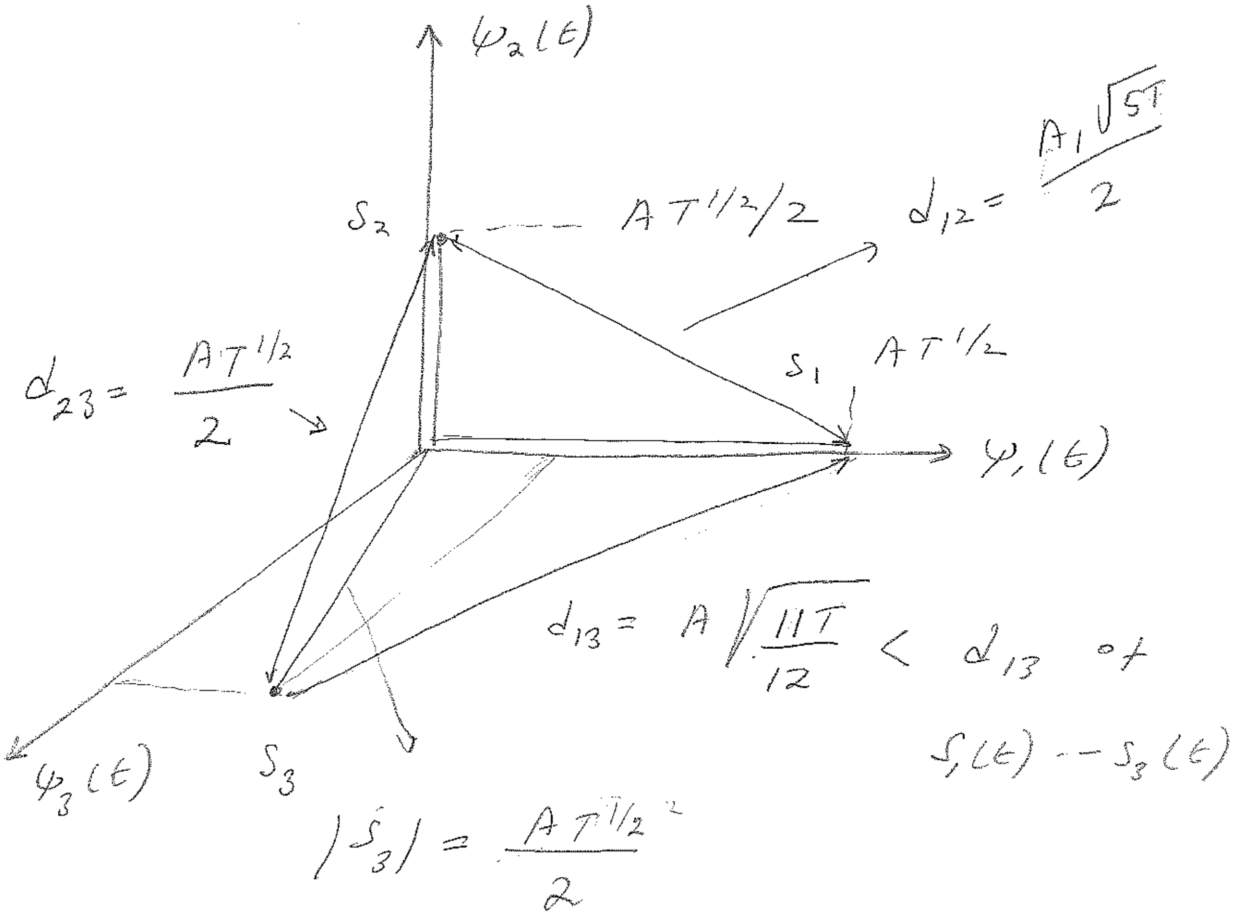
$$+ \frac{A}{2}\sqrt{\frac{2}{T}}\int_{T/3}^{2T/3} dt + \frac{A}{2}\frac{1}{\sqrt{2T}}\int_{2T/3}^{T} dt = \frac{A}{3}\sqrt{2T}$$

So

$$s_3'(t) = \frac{AT^{1/2}}{6}\varphi_1'(t) + \frac{A}{3}\sqrt{2T}\,\varphi_3'(t) \quad \text{and}$$

$$s_3' = [\,AT^{1/2}/6 \quad 0 \quad A\sqrt{2T}/3\,]$$

So the constellation for $s_1'(t) \cdots s_3'(t)$



So if we compare the constellation of $s_1(t) -- s_3(t)$ with that of $s_1'(t) -- s_3'(t)$ we find

1) Both case are three dimensional and use the same total energy or same average energy

2) In case of $s_1'(t) --- s_3'(t)$ less band-width is used since T duration is sliced into 3, whereas in $s_1(t) --- s_3(t)$, it is 4

3) In $s_1'(t) --- s_3'(t)$ d_{13} is less than d_{13} of $s_1(t) -- s_3(t)$, so we expect

$$P_{e[s_1(t) -- s_3(t)]} < P_{e[s_1'(t) -- s_3'(t)]}$$

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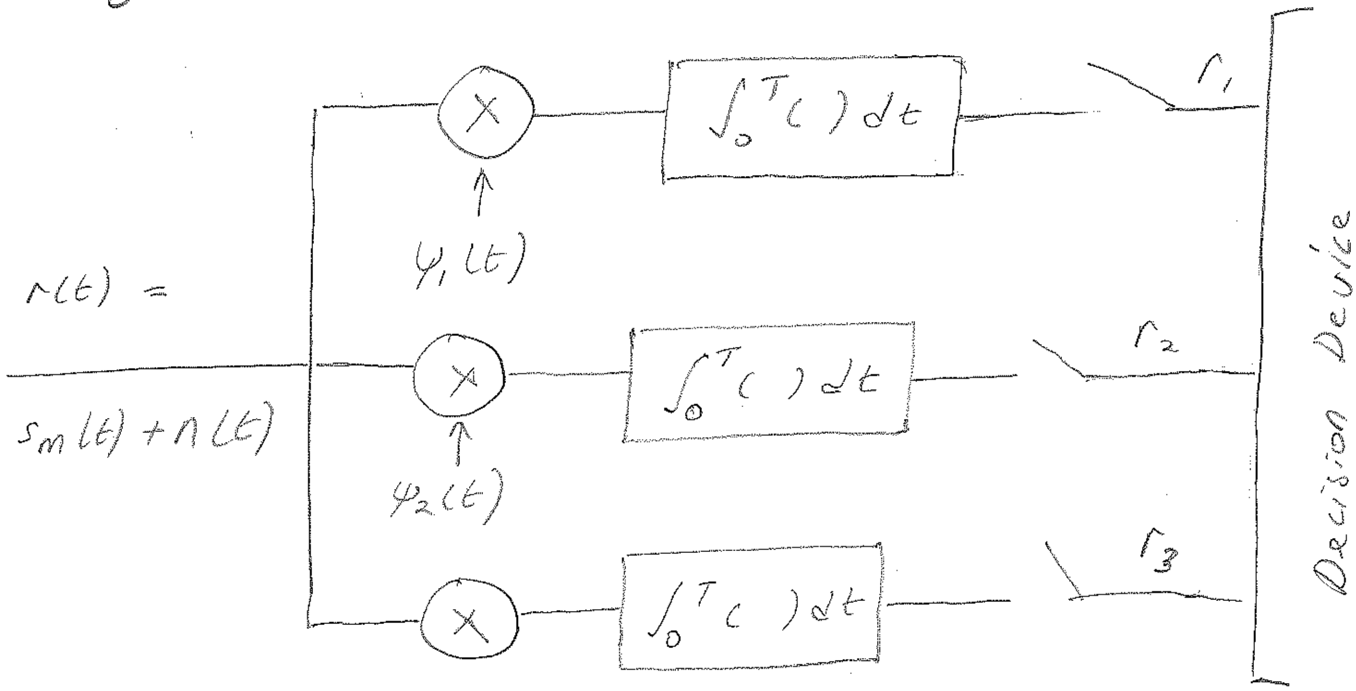
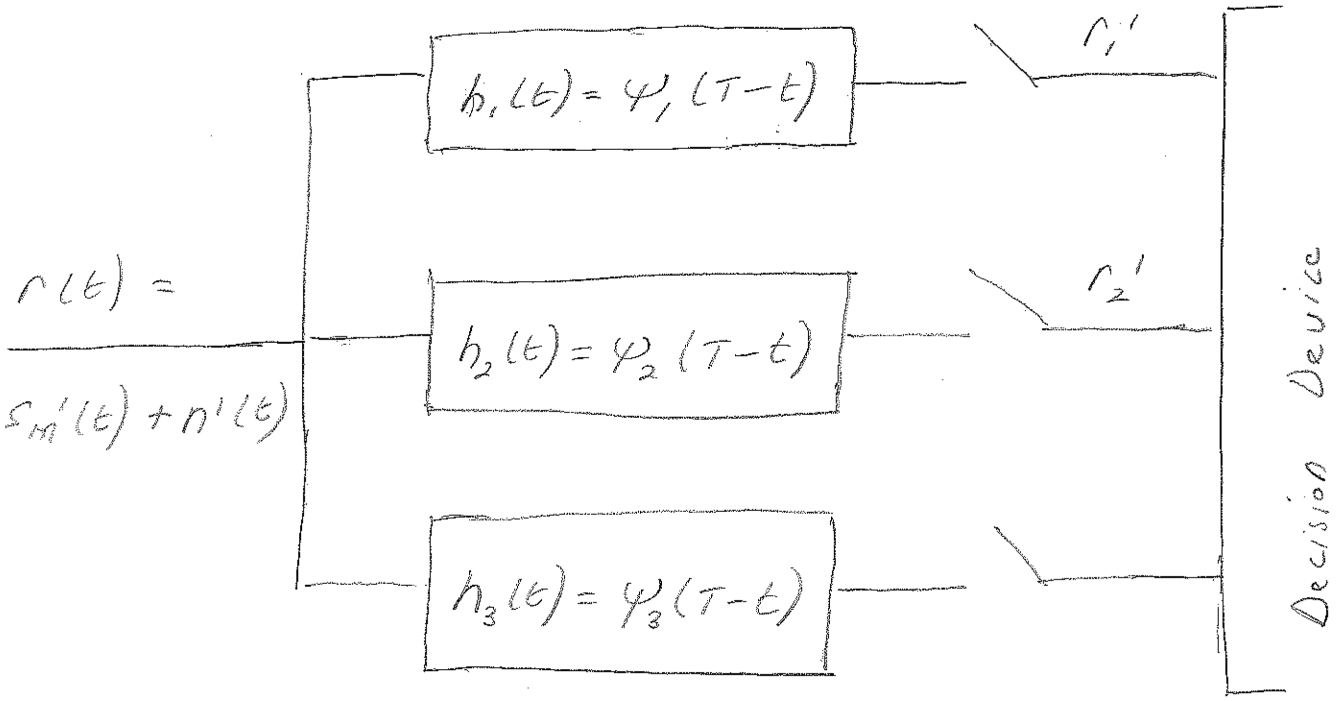
For both cases (receiver) correlator and MF block diagrams are shown below

Correlator





MF

If $s_1(t)$ is transmitted then both correlator and MF output is

$$r_1 = A\,T^{1/2} + n_1 \;,\quad r_2 = n_2 \;,\quad r_3 = n_3$$

If $s_1'(t)$ is transmitted, both correlator and MF output is

$$r_1' = A\,T^{1/2} + n_1' \;,\quad r_2' = n_2' \;,\quad r_3' = n_3'$$

2. (30 Points) Answer the following questions as **True** or **False**. For the **False** ones give the correct answer or the reason. For the **True** ones justify your answer.

a) Probability of error increases when *M*ary levels of the signal increase : True

with increase in Mary level, if binary input is kept constant, symbol duration will increase so energies of signal vectors will increase, but this will not compensate for the reductions in the min distance, so $M\uparrow$ $P_e\uparrow$

b) PSK and ASK are two dimensional :

Partially true, partially false since PSK is two dimensional but ASK is one dimensional

c) By rotating constellation diagram, we obtain better probability of error performance : False

rotation of constellation diagram simply redefines the position of signal vectors with respect to orthonormal basis functions. It is the relative distance which contributes to P_e

d) We do not use QAM for *M*ary values less that 16 : Generally true, since 4QAM = 4PSK and 8QAM does not bring much advantage compared to 8PSK. But from 16QAM onwards, we can get some P_e improvement compared with the same Mary PSK.

e) FM has better noise performance since the bandwidth of the message signal is expanded during modulation : True, it is the expansion (spreading) of message signal and its fold back to the original spectrum (at receiver) and correlation between the signal components and the absence of correlation between noise components that give rise to better noise performance in FM.