# Çankaya University – ECE Department – ECE 376

Student Name : Open source exam
Student Number : Duration : 2 hours

Questions

1. (35 Points) An OFDM system is based on 16 QAM, where the bit rate is 5 Mbits/sec. Find the appropriate subcarriers of this OFDM system with different options, specifying the minimum and maximum numeric frequency values that can be assigned to these subcarriers. Show that these subcarriers are orthogonal along time axis. Draw the approximate frequency spectrum of the modulated subcarriers.

   Assuming a rectangular constellation is used for the 16 QAM signal set and further assuming that the signal vector with the smallest length has unit energy (i.e. 1), find the energies of other signal vectors in terms of the length of the smallest signal vector. Comment how probability of error would change for signals placed at different positions in this 16 QAM constellation.

**Solution :** Since bit rate, i.e. $1/T_b$ is 5 Mbits/sec, then $T_s = T_b \times \log_2 M = 4/5$ .

In this case any two subcarriers $f_k$ and $f_j$ can be arranged such that $f_k - f_j = \frac{n}{T}$, where $T = T_s$ or $T = T_s \times M = 16T_s$, assuming the number of subcarriers is made equal to number of symbols in 16 QAM signal set. Ensuring that the first subcarrier $(f_1)$ has at least one complete cycle (which means, $n = 1$) within $T = 16T_s = 12.8$ , then

$f_1 = \frac{1}{12.8} = \frac{5}{}$ MHz $= 78.125$ kHz. Note that $f_1 = 78.125$ kHz is the minimum, we can also choose $f_1$ to be integer multiples of $78.125$ kHz up to infinity, then of course, the OFDM spectrum would unnecessarily be wider.

Adopting $f_1 = 78.125$ kHz , then a complete list of OFDM subcarriers would be as follows

$$f_1 = 78.125 \text{ kHz}, f_2 = 156.25 \text{ kHz}, f_3 = 234.375 \text{ kHz}, f_4 = 312.5 \text{ kHz}, f_5 = 390.625 \text{ kHz},$$
$$f_6 = 468.75 \text{ kHz}, f_7 = 546.875 \text{ kHz}, f_8 = 625 \text{ kHz}, f_9 = 703.125 \text{ kHz}, f_{10} = 781.25 \text{ kHz},$$
$$f_{11} = 859.375 \text{ kHz}, f_{12} = 937.5 \text{ kHz}, f_{13} = 1015.625 \text{ kHz}, f_{14} = 1093.75 \text{ kHz}, f_{15} = 1171.875 \text{ kHz},$$
$$f_{16} = 1250 \text{ kHz}$$

The above list is the minimum of subcarrier frequency numeric values. Any integer multiples (by keeping the frequency spacing of $78.125$ kHz or raising it by integer multiples) of the above would also serve as subcarriers.

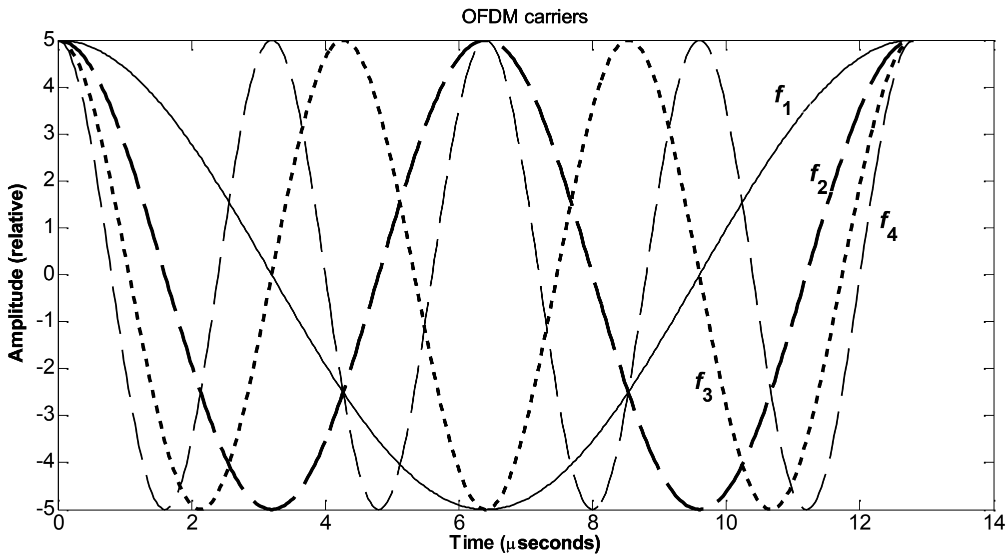
To show the orthogonality, we arbitrarily take third and tenth subcarriers in exponential forms and perform the following integration, which represents the matched filter detection of the OFDM receiver,

$$\int_0^{12.8\times10^{-6}} \exp\left(j2\pi\times234375t\right)\exp\left(-j2\pi\times781250t\right)dt = \int_0^{12.8\times10^{-6}} \exp\left(-j2\pi\times546875t\right)dt =$$

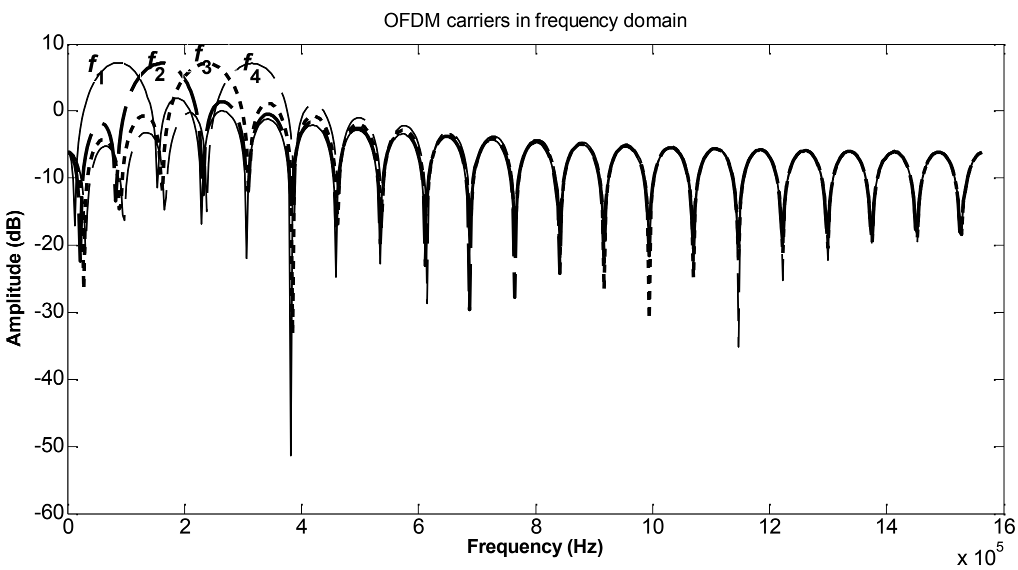
$$\frac{-1}{2\pi j\times546875}\exp\left(-j2\pi\times546875t\right)\Big|_0^{12.8\times10^{-6}} = \frac{-1}{2\pi j\times546875}\left[\exp\left(-j2\pi\times7\right)-\exp\left(0\right)\right]=0$$

The above is actually in the form of sum of all subcarriers (modulated by one of the 16 QAM symbols) times a specific subcarrier of that branch. To this end, the above would only give a result different from zero when the two subcarriers are at the same frequency, which is the expected operation of the OFDM demodulator.

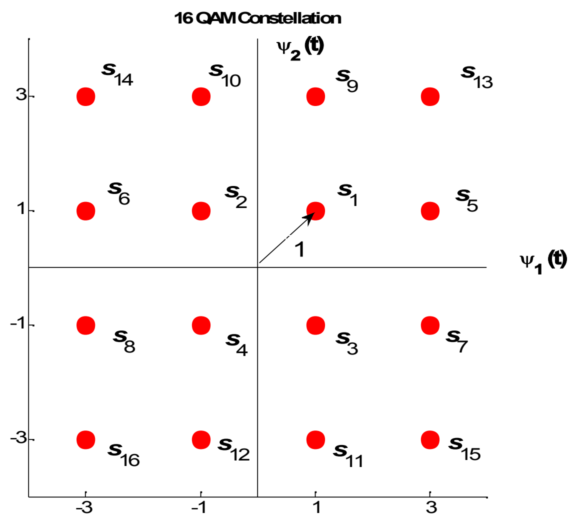
Time waveforms of the first four subcarriers are shown below ;




Again the frequency spectrum for the first subcarriers are also shown below ;



The 16 rectangular QAM constellation described in the question is depicted below ;




We considere that 16 QAM constellation consists of two squares and two rectancles, where the signal vector ends are located on the vertices of these squares and rectangles. This the smaller (inner) square contains the signal vectores of $s_1,\ s_2,\ s_3,\ s_4$ and the larger (outer) one square contains the signal vectors, $s_{13},\ s_{14},\ s_{15},\ s_{16}$. The two remaining rectangles have $s_5,\ s_6,\ s_7,\ s_8$ and $s_9,\ s_{10},\ s_{11},\ s_{12}$. If the lengths of $s_1,\ s_2,\ s_3,\ s_4$ are all equal and unity, i.e. 1, then the sides of the inner square (also distance between all adjacent vectors) are also all equal to $\sqrt{2}$. Using geometric relationships, we find that lengths of other signal vectors are as follows ;

Lengths of $s_5,\ s_6,\ s_7,\ s_8$ and $s_9,\ s_{10},\ s_{11},\ s_{12}$ are all equal to $\sqrt{5}$,
Lengths of $s_{13},\ s_{14},\ s_{15},\ s_{16}$ are all equal to 3.

Thus the energies of different signal vectors will be

Energies of $s_1,\ s_2,\ s_3,\ s_4$ are all equal to 1,
Energies of $s_5,\ s_6,\ s_7,\ s_8$ and $s_9,\ s_{10},\ s_{11},\ s_{12}$ are all equal to 5,
Energies of $s_{13},\ s_{14},\ s_{15},\ s_{16}$ are all equal to 9.

It is known that probability of error is inversely proportional to signal energy, also inversely proportional to distance between signal vector ends. Further note that, signal vectors on the smaller (inner) square have adjacent signal vectors (at the smallest distance of $\sqrt{2}$) to all four $90^0$ directions, the signal vectors on the vertices of the two rectangles have adjacent vectors to only three of the $90^0$ directions, finally the signal vectors on the larger (outer) square have adjacent vectors to only two of the $90^0$ directions. Bearing these points in mind, we see that

$$P_e(s_{13}) = P_e(s_{14}) = P_e(s_{15}) = P_e(s_{16}) < P_e(s_5) = P_e(s_6) = P_e(s_7) = P_e(s_8)$$
$$= P_e(s_9) = P_e(s_{10}) = P_e(s_{11}) = P_e(s_1) < P_e(s_2) = P_e(s_3) = P_e(s_4)$$

2. (35 Points) In a spread spectrum system, for two message signals, two PN sequences ${}_1c(t)$ and ${}_2c(t)$ of length $L_c = 15$ are used. The vectorial representation of these PN sequences $({}_1C$ and ${}_2C)$ are given below. Are ${}_1c(t)$ and ${}_2c(t)$ maximum length codes ?. Find the time shifted autocorrelation and cross correlations of ${}_1c(t)$ and ${}_2c(t)$ and plot them.

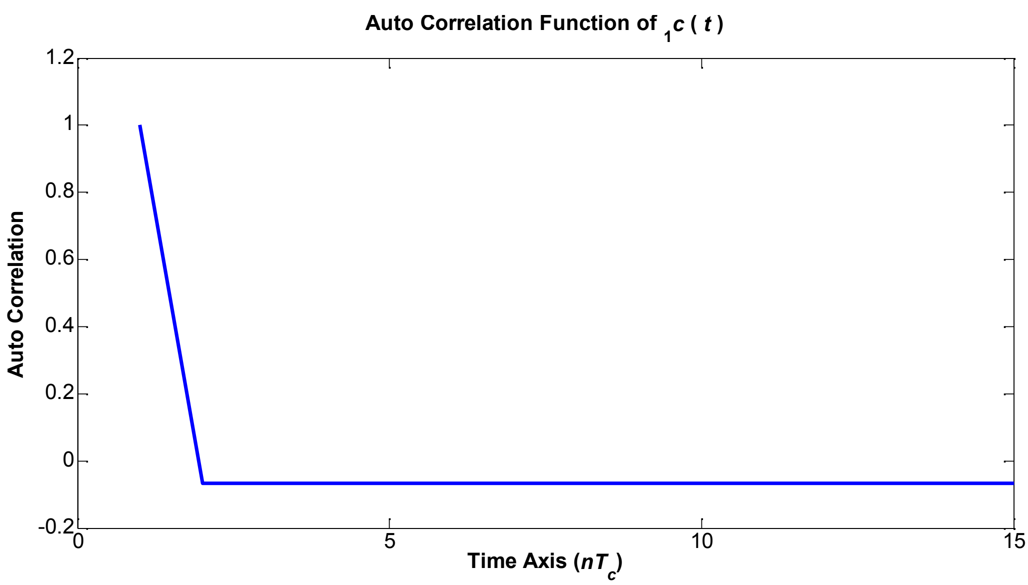
$$
\begin{aligned}
{}_1C &= [-1\ -1\ -1\ 1\ -1\ -1\ 1\ 1\ -1\ 1\ -1\ 1\ 1\ 1\ 1] \\
{}_2C &= [-1\ -1\ -1\ 1\ 1\ 1\ 1\ -1\ 1\ -1\ 1\ 1\ -1\ -1\ 1]
\end{aligned}
$$

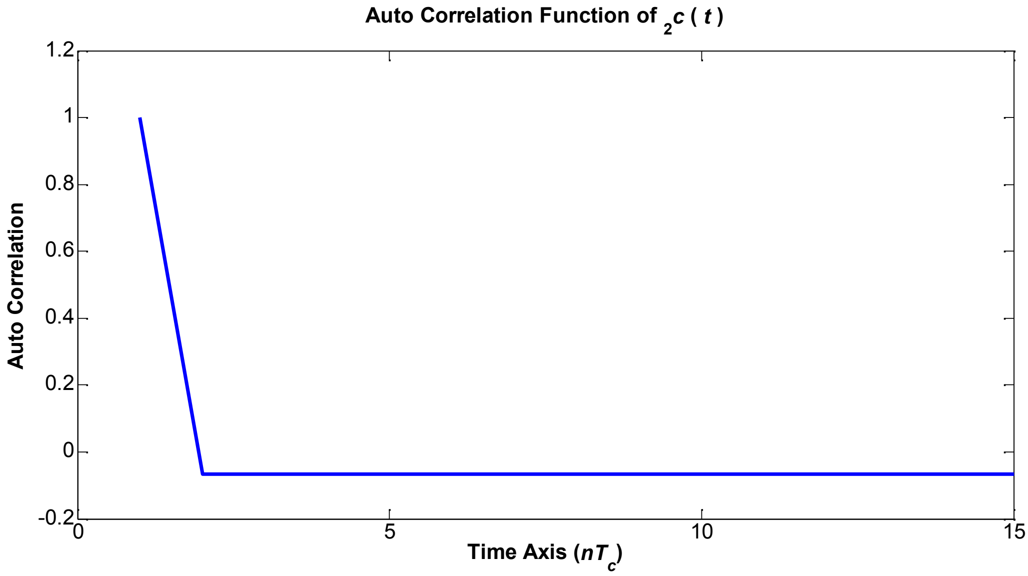
If ${}_1c(t)$ and ${}_2c(t)$ are used to spread the message signals $v_1(t)$ and $v_2(t)$ each with one $T_b = 1$ msec duration, plot the resulting spread waveforms. At the receiver, estimate how much interference will result when demodulating (dispreading). Draw the relevant waveforms and spectrums.

**Solution :** From the givens, it is clear that $T_c = T_b/15 = 1/15$ msec, $L_c = T_b/T_c = 15$. For the time shifted (cyclic) auto correlations of ${}_1c(t)$ and ${}_2c(t)$, we have

$$
\begin{aligned}
&\sum_{n=0}^{L_c-1} {}_1c(t) \bullet_1 c'(t-nT_c)/L_c = \sum_{n=0}^{14} {}_1c(t) \bullet_1 c'(t-nT_c)/15 \\
&= [1\ -1/15\ -1/15\ -1/15\ -1/15\ -1/15\ -1/15\ -1/15\ -1/15 \\
&\quad -1/15\ -1/15\ -1/15\ -1/15\ -1/15\ -1/15] \\
&\sum_{n=0}^{L_c-1} {}_2c(t) \bullet_2 c'(t-nT_c)/L_c = \sum_{n=0}^{14} {}_2c(t) \bullet_2 c'(t-nT_c)/15 \\
&= [1\ -1/15\ -1/15\ -1/15\ -1/15\ -1/15\ -1/15\ -1/15\ -1/15 \\
&\quad -1/15\ -1/15\ -1/15\ -1/15\ -1/15\ -1/15]
\end{aligned}
$$

Graphically we obtain the following

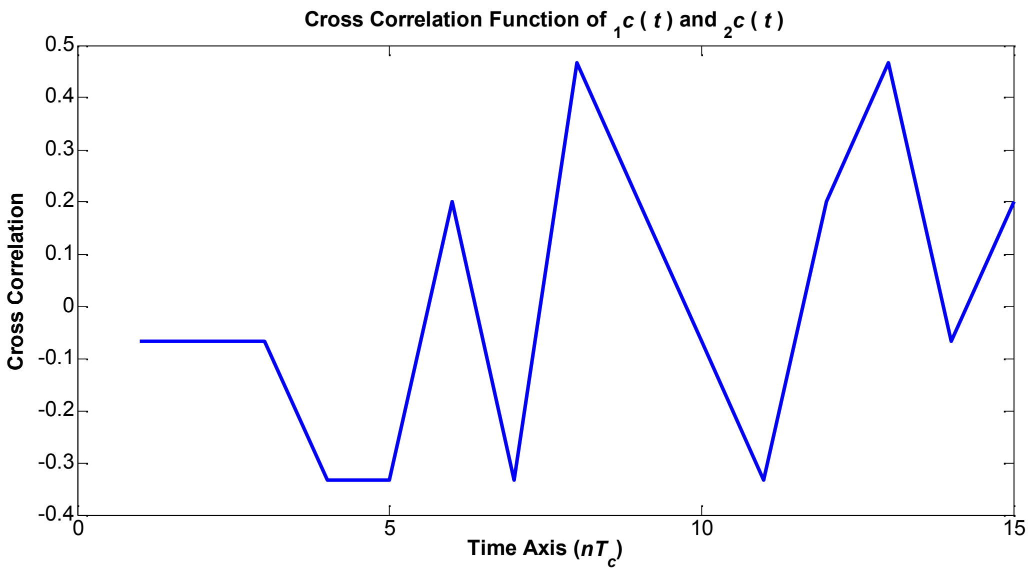





So both $_1c(t)$ and $_2c(t)$ are maximum length PN codes. To find the time shifted cross correlation of $_1c(t)$ and $_2c(t)$, we use the same computation as above, hence

$$\sum_{n=0}^{L_c-1} {}_1c(t) \bullet_2 c'(t-nT_c)/L_c = \sum_{n=0}^{14} {}_1c(t) \bullet_2 c'(t-nT_c)/15$$
$$= [-1/15 \ \ -1/15 \ \ -1/15 \ \ -1/3 \ \ -1/3 \ \ 1/5 \ \ -1/3 \ \ 7/15 \ \ 1/5$$
$$-1/15 \ \ -1/3 \ \ 1/5 \ \ 7/15 \ \ -1/15 \ \ 1/5]$$

Graphically this means

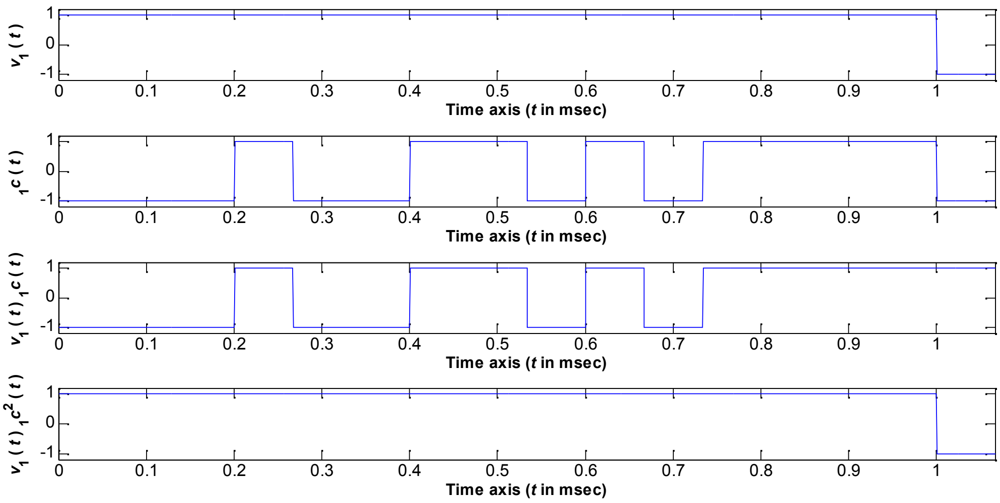



At receiver during dispreading operation, we perform the following integration to demodulate the message (bit which is either + 1 or - 1) carried by PN sequence of $_1c(t)$

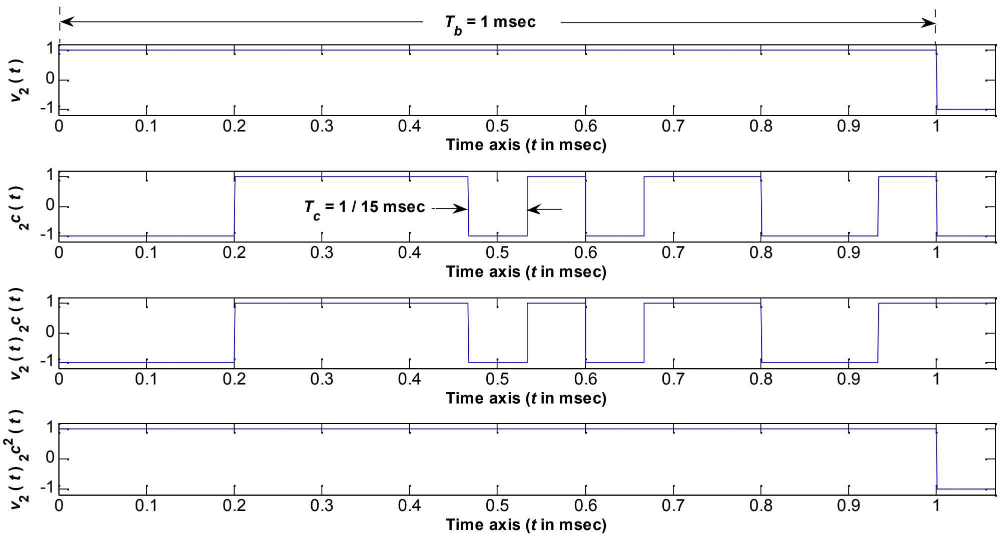
$$\frac{1}{L_c}\int_0^{T_b}\left[v_1(t)\,{}_1c(t)+v_2(t)\,{}_2c(t)\right]{}_1c(t)\,dt = 1\pm\frac{1}{15}$$

Here 1 represents the successfully demodulated message bit belonging to ${}_1c(t)$, whereas $\frac{1}{15}$ is the (normalized) interference term.

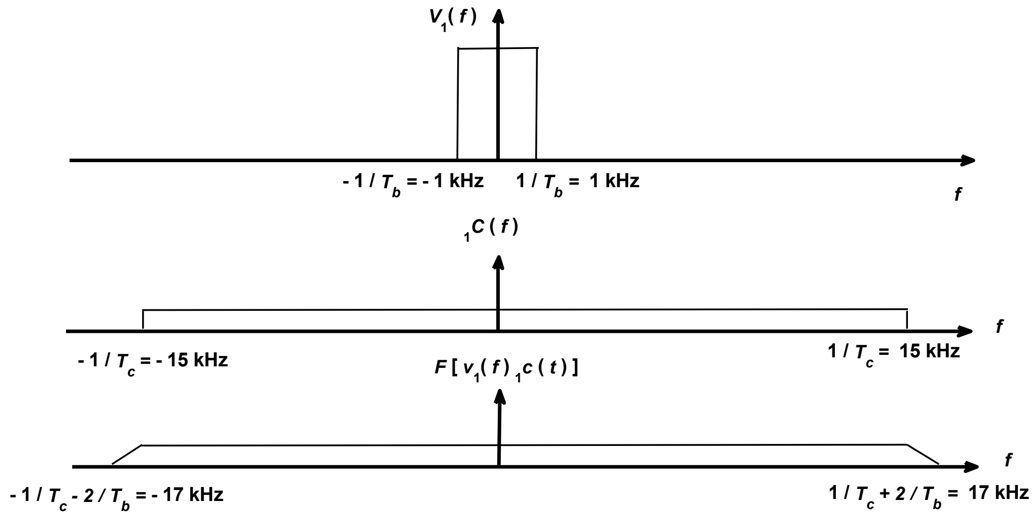
Time waveforms for $v_1(t)$ are shown below




Time waveforms for $v_2(t)$ are shown below




The frequency spectrums are shown below, with the one for $v_2(t)$ being the same


$V_1(f)$
$-1/T_b = -1$ kHz
$1/T_b = 1$ kHz
$f$
$_1C(f)$
$f$
$-1/T_c = -15$ kHz
$1/T_c = 15$ kHz
$F[v_1(f)\,_1c(t)]$
$f$
$-1/T_c - 2/T_b = -17$ kHz
$1/T_c + 2/T_b = 17$ kHz

3. (30 Points) Answer the following questions as **True** or **False.** For the **False** ones give the correct answer or the reason. For the **True** ones justify your answer.

a) PN sequences are used for spreading and dispreading operations : True, the same PN sequence must be used both for spreading and dispreading one particular message signal.

b) In a PSK radio, noise power is determined by looking at the *M*ary value : Essentially this becomes true, since noise power is calculated from $P_n = kTB$, where $k$ is Boltzmann's constant, $T$ is the absolute temperature in degrees Kelvin, $B$ is bandwidth, which is inversely proportional to *M*ary value.

c) As bit rate increases, probability of error increases as well : True, if all other parameters are kept the same, then increasing bit rate means reducing bit duration (and consequently symbol duration), thus reducing energy. Another way of looking at it is that as bit rate increases, bandwidth also increase, thus noise power increases.

d) OFDM is used for noisy channels : False, the main idea in OFDM is to overcome the communication channel irregular response (which is the result of multipath fading effects), by dividing the whole spectrum into narrow slices.

e) In spread spectrum systems, there is no protection against interference : False, on the contrary, spread spectrum signals share the time axis and the same frequency spectrum, therefore the PN sequences are selected such that there is minimum cross correlation (or minimum interference in this case) between them.