

Çankaya University – ECE Department – ECE 376 (MT)

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Open Source Exam

Questions

1. (70 Points) For the time waveforms of the signal set, $s_1(t) \cdots s_4(t)$ given in Fig. 1.1, perform the following operations.
- Identify the type of modulation and dimensionality in this signal set. Write mathematical expression for $s_1(t) \cdots s_4(t)$ and the corresponding basis functions, $\psi_1(t) \cdots \psi_N(t)$ and plot $\psi_1(t) \cdots \psi_N(t)$. Write for the signal vectors $\mathbf{s}_1 \cdots \mathbf{s}_4$, and plot the corresponding constellation diagram. Find the distance between signal vector ends.
 - Draw the demodulator as correlator and matched filter. Assuming that the signal $s_1(t)$ from constellation is transmitted, find the outputs of the correlator and matched filter.
 - Find the probability of error and decision regions via the evaluations of correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$ again assuming $s_1(t)$ was transmitted. Comment on whether you would find any probability of error performance difference if another signal from the set $s_2(t) \cdots s_4(t)$ was transmitted.

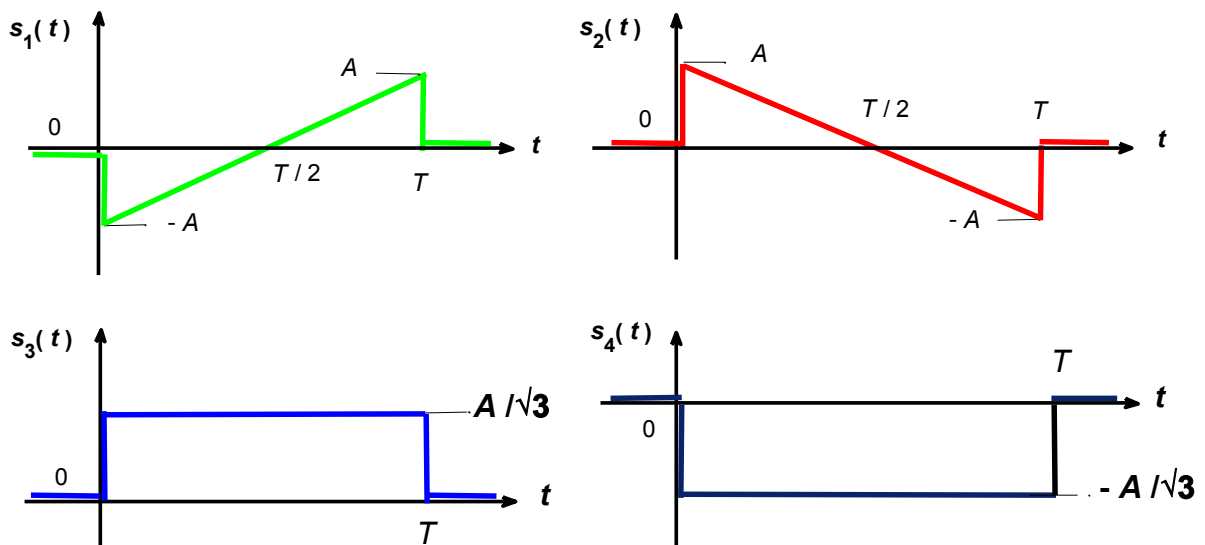


Fig. 1.1 The time waveforms, $s_1(t) \cdots s_4(t)$ for Q1.

Solution : a. The given time waveforms $s_1(t) \cdots s_4(t)$ will constitute 4 PSK, if the energies of signals are equal, if they can be represented by two orthonormal basis functions. For these tests we write

$$\begin{aligned}\varepsilon_{s_1} &= \int_{-\infty}^{\infty} s_1^2(t) dt = \int_0^T s_1^2(t) dt = A^2 \int_0^T \left(\frac{2t}{T} - 1\right)^2 dt = \frac{A^2 T}{3} = \varepsilon_{s_2} \\ \varepsilon_{s_3} &= \int_{-\infty}^{\infty} s_3^2(t) dt = \int_0^T s_3^2(t) dt = \frac{A^2}{3} \int_0^T dt = \frac{A^2 T}{3} = \varepsilon_{s_4}\end{aligned}\quad (1.1)$$

With eye inspection, we can devise the following orthonormalized basis functions for the signal set, $s_1(t) \cdots s_4(t)$

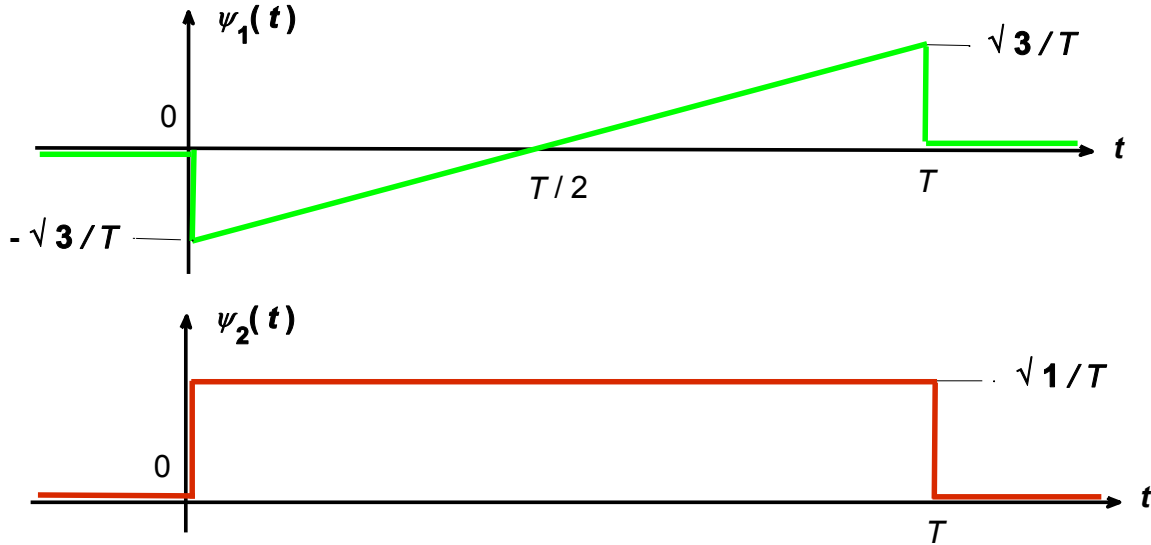


Fig. 1.2 Orthonormalized basis functions, $\psi_1(t)$, $\psi_2(t)$ for the signal set, $s_1(t) \cdots s_4(t)$.

The time waveform expressions for $\psi_1(t)$, $\psi_2(t)$ are

$$\psi_1(t) = \begin{cases} \sqrt{3/T} \left(\frac{2t}{T} - 1\right) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}, \quad \psi_2(t) = \begin{cases} \sqrt{1/T} & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}\quad (1.2)$$

It is easy to verify that

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_1(t) \psi_2(t) dt &= \int_0^T \psi_1(t) \psi_2(t) dt = 0 \\ \int_{-\infty}^{\infty} \psi_1^2(t) dt &= \int_0^T \psi_1^2(t) dt = \int_{-\infty}^{\infty} \psi_2^2(t) dt = \int_0^T \psi_2^2(t) dt = 0\end{aligned}\quad (1.3)$$

The time waveform expression for $s_1(t) \cdots s_4(t)$ and in terms of $\psi_1(t)$, $\psi_2(t)$, the signal vector representations for $\mathbf{s}_1 \cdots \mathbf{s}_4$ are given in (1.4). The corresponding constellation diagram is provided in Fig. 1.3.

$$\begin{aligned}
s_1(t) &= \begin{cases} A\left(\frac{2t}{T}-1\right) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}, \quad s_2(t) = \begin{cases} A\left(1-\frac{2t}{T}\right) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \\
s_3(t) &= \begin{cases} A/\sqrt{3} & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}, \quad s_4(t) = \begin{cases} -A/\sqrt{3} & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \\
s_1(t) &= A\sqrt{T/3}\psi_1(t), \quad s_2(t) = -A\sqrt{T/3}\psi_1(t), \quad s_3(t) = A\sqrt{T/3}\psi_2(t), \quad s_4(t) = -A\sqrt{T/3}\psi_2(t) \\
\mathbf{s}_1 &= [s_{11}, s_{12}] = [A\sqrt{T/3}, 0], \quad \mathbf{s}_2 = [s_{21}, s_{22}] = [-A\sqrt{T/3}, 0] \\
\mathbf{s}_3 &= [s_{31}, s_{32}] = [0, A\sqrt{T/3}], \quad \mathbf{s}_4 = [s_{41}, s_{42}] = [0, -A\sqrt{T/3}] \\
d_{12} &= d_{34} = 2A\sqrt{T/3} = 2\sqrt{\varepsilon_s}, \quad d_{13} = d_{24} = A\sqrt{2T/3} = \sqrt{2\varepsilon_s} \\
|\mathbf{s}_1| &= |\mathbf{s}_2| = |\mathbf{s}_3| = |\mathbf{s}_4| = A\sqrt{T/3}, \quad \varepsilon_s = \frac{A^2 T}{3}
\end{aligned} \tag{1.4}$$

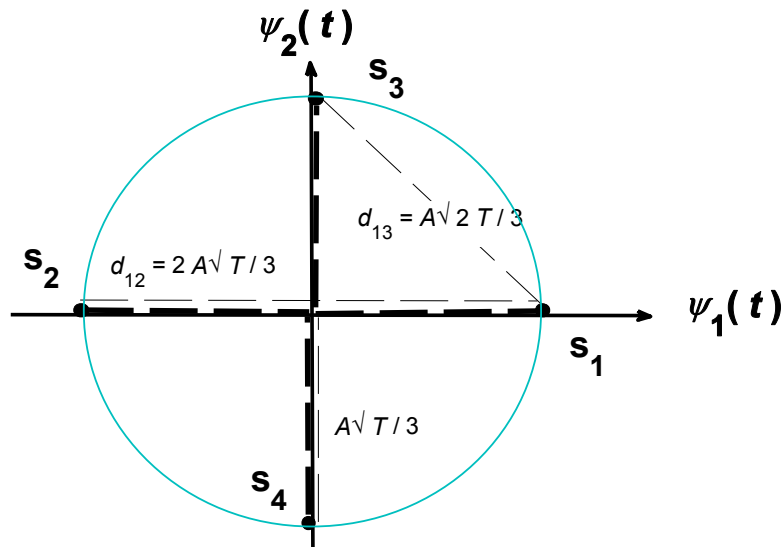


Fig. 1.3 Constellation diagram for the signal set, $s_1(t) \dots s_4(t)$.

b. The block diagrams of demodulator as correlator and matched filter are given in Figs. 6.7a and 6.7b of ECE376_Dimensionality of Signals_ASK_PSK_QAM_FSK_Jan 2013_HTE. They are not repeated here to save space.

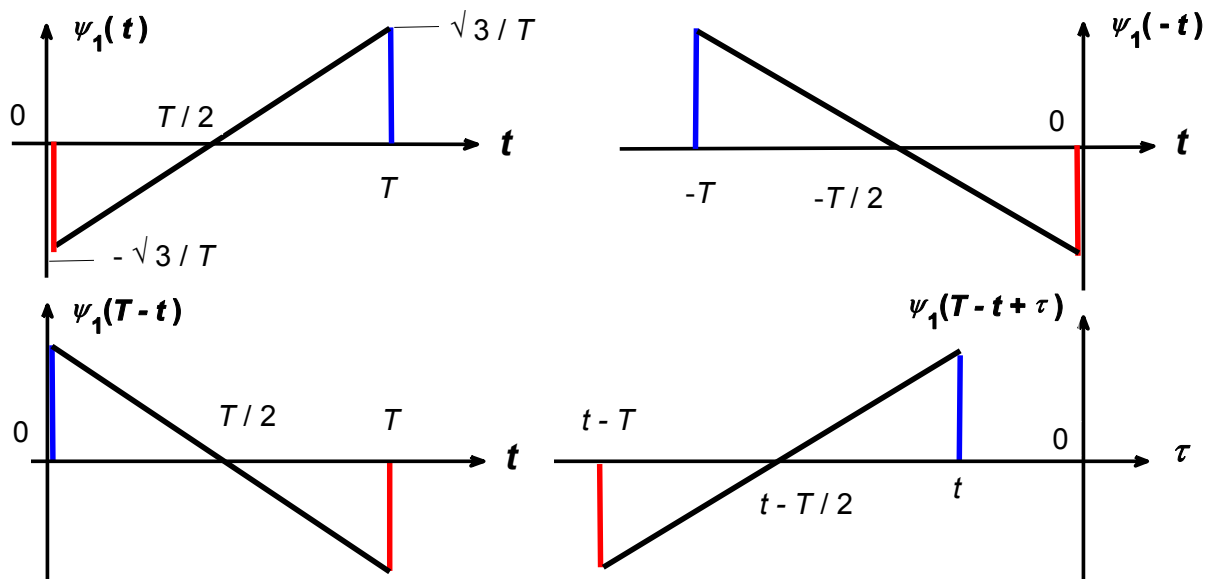


Fig. 1.4 Orientation of orthonormalized basis function, $\psi_1(t)$ for the convolution integral.

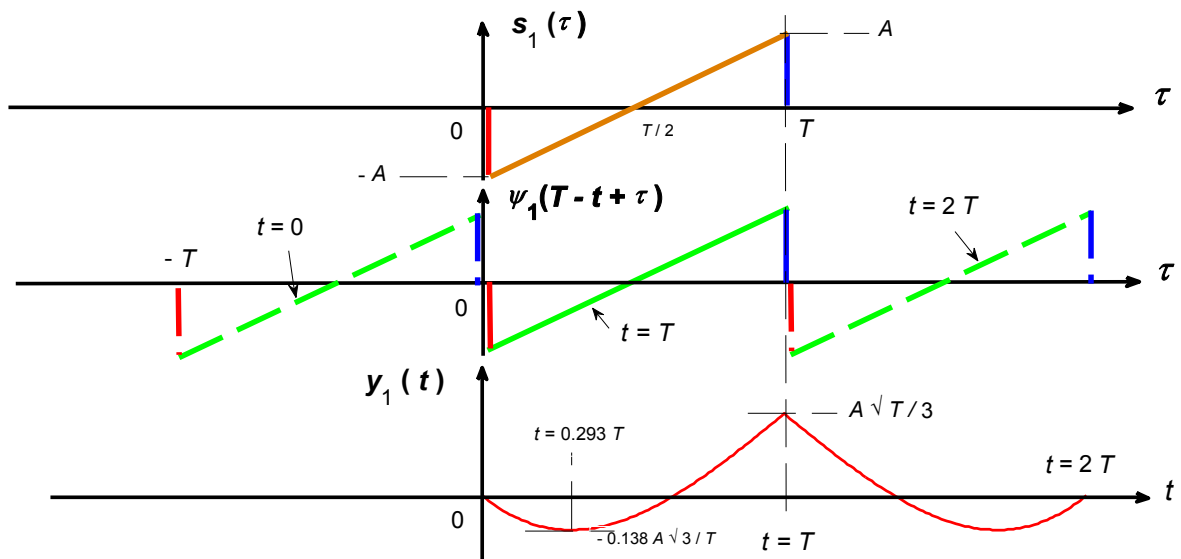


Fig. 1.5 Graphical illustration of the convolution operation implemented in the upper matched filter of 4 PSK demodulator.

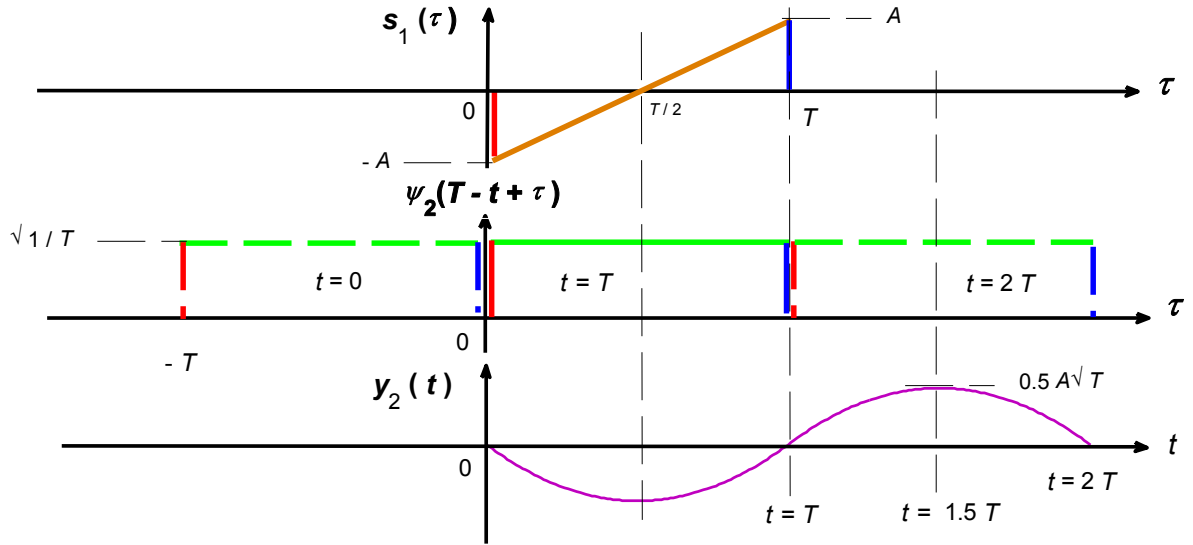


Fig. 1. 6 Graphical illustration of the convolution operation implemented in the lower matched filter of 4 PSK demodulator.

Taking into account (1.2), (1.4) and Fig. 1.4, we write

$$s(\tau) = A \left(\frac{2\tau}{T} - 1 \right), \quad \psi_1(T-t+\tau) = \sqrt{3/T} \left[\frac{2(T-t+\tau)}{T} - 1 \right], \quad \psi_2(T-t+\tau) = \sqrt{3/T} \quad (1.5)$$

Then from Fig. 1.5, we get

$$y_1(t) = \begin{cases} y_{11}(t) = 0 & t \leq 0 \\ y_{12}(t) = \int_0^t s_1(\tau) \psi_1(T-t+\tau) d\tau = A\sqrt{3/T} \int_0^t \left(\frac{2\tau}{T} - 1 \right) \left[\frac{2(T-t+\tau)}{T} - 1 \right] d\tau & 0 \leq t \leq T \\ y_{13}(t) = \int_{t-T}^T s_1(\tau) \psi_1(T-t+\tau) d\tau = A\sqrt{3/T} \int_{t-T}^T \left(\frac{2\tau}{T} - 1 \right) \left[\frac{2(T-t+\tau)}{T} - 1 \right] d\tau & T \leq t \leq 2T \\ y_{14}(t) = 0 & t \geq 2T \end{cases} \quad (1.6)$$

Performing the integrations with the help of m file named "ECE376_MT_2014_Q1", we have

$$y_1(t) = \begin{cases} y_{12}(t) = A\sqrt{3/T} \left(-\frac{2t^3}{3T^2} + \frac{2t^2}{T} - t \right) & 0 \leq t \leq T \\ y_{13}(t) = A\sqrt{3/T} \left(\frac{2t^3}{3T^2} - \frac{2t^2}{T} + t + \frac{2T}{3} \right) & T \leq t \leq 2T \end{cases} \quad (1.7)$$

It is possible to verify that

$$\begin{aligned}
y_{12}(t=0) &= 0 \\
y_{12}(t=T) &= A\sqrt{T/3} = y_{13}(t=T) \\
y_{13}(t=2T) &= 0
\end{aligned} \tag{1.8}$$

$y_1(t)$ is plotted in Fig. 1.5. Doing the same for $y_2(t)$, we get

$$y_2(t) = \begin{cases} y_{21}(t) = 0 & t \leq 0 \\ y_{22}(t) = \int_0^t s_1(\tau) \psi_2(T-t+\tau) d\tau = A\sqrt{1/T} \int_0^t \left(\frac{2\tau}{T}-1\right) d\tau = A\sqrt{1/T} \left(\frac{t^2}{T}-t\right) & 0 \leq t \leq T \\ y_{23}(t) = \int_{t-T}^T s_1(\tau) \psi_2(T-t+\tau) d\tau = A\sqrt{1/T} \int_{t-T}^T \left(\frac{2\tau}{T}-1\right) d\tau = A\sqrt{1/T} \left(-\frac{t^2}{T}+3t-2T\right) & T \leq t \leq 2T \\ y_{24}(t) = 0 & t \geq 2T \end{cases} \tag{1.9}$$

From (1.9) and Fig. 1.6, we see that

$$\begin{aligned}
y_{22}(t=0) &= 0 \\
y_{22}(t=T) &= 0 = y_{13}(t=T) \\
y_{13}(t=2T) &= 0
\end{aligned} \tag{1.10}$$

c. At the sampling instance of $t=T$, from (1.8) and (1.10), we get

$$\begin{aligned}
r_1 &= \int_0^T r(t) \psi_1(t) dt = \int_0^T s_1(t) \psi_1(t) dt + \int_0^T n(t) \psi_1(t) dt \\
&= A\sqrt{T/3} + n_1, \quad n_1 = \int_0^T n(t) \psi_1(t) dt \\
r_2 &= \int_0^T r(t) \psi_2(t) dt = \int_0^T s_1(t) \psi_2(t) dt + \int_0^T n(t) \psi_2(t) dt \\
&= 0 + n_2, \quad n_2 = \int_0^T n(t) \psi_2(t) dt, \quad \mathbf{r} = [r_1; r_2] = \begin{bmatrix} A\sqrt{T/3} + n_1 \\ n_2 \end{bmatrix}
\end{aligned} \tag{1.11}$$

Carrying out the correlation metrics calculations, we will obtain the results stated in (1.12), where we have simplified $C(\mathbf{r}, \mathbf{s}_m)$, eliminating all common terms that will have no effect on our decision boundaries.

$$\begin{aligned}
m=1, C(\mathbf{r}, \mathbf{s}_1) &= \mathbf{s}_1 \cdot \mathbf{r} = \left[A\sqrt{T/3}, 0 \right] \begin{bmatrix} A\sqrt{T/3} + n_1 \\ n_2 \end{bmatrix} = \frac{A^2T}{3} + An_1\sqrt{\frac{T}{3}} \\
m=2, C(\mathbf{r}, \mathbf{s}_2) &= \mathbf{s}_3 \cdot \mathbf{r} = \left[-A\sqrt{T/3}, 0 \right] \begin{bmatrix} A\sqrt{T/3} + n_1 \\ n_2 \end{bmatrix} = -\frac{A^2T}{3} - An_1\sqrt{\frac{T}{3}} \\
m=3, C(\mathbf{r}, \mathbf{s}_3) &= \mathbf{s}_2 \cdot \mathbf{r} = \left[0, A\sqrt{T/3} \right] \begin{bmatrix} A\sqrt{T/3} + n_1 \\ n_2 \end{bmatrix} = An_2\sqrt{\frac{T}{3}} \\
m=4, C(\mathbf{r}, \mathbf{s}_4) &= \mathbf{s}_4 \cdot \mathbf{r} = \left[0, -A\sqrt{T/3} \right] \begin{bmatrix} A\sqrt{T/3} + n_1 \\ n_2 \end{bmatrix} = -An_2\sqrt{\frac{T}{3}}
\end{aligned} \tag{1.12}$$

Now for correct decision, the following three conditions must be satisfied

$$\begin{aligned}
C(\mathbf{r}, \mathbf{s}_1) > C(\mathbf{r}, \mathbf{s}_2) &: \frac{A^2T}{3} + An_1\sqrt{\frac{T}{3}} > -\frac{A^2T}{3} - An_1\sqrt{\frac{T}{3}} \rightarrow \frac{\sqrt{\epsilon_s}}{A\sqrt{T/3}} > -n_1 \\
C(\mathbf{r}, \mathbf{s}_1) > C(\mathbf{r}, \mathbf{s}_3) &: \frac{A^2T}{3} + An_1\sqrt{\frac{T}{3}} > An_2\sqrt{\frac{T}{3}} \rightarrow A\sqrt{T/3} + n_1 > n_2 \\
C(\mathbf{r}, \mathbf{s}_1) > C(\mathbf{r}, \mathbf{s}_4) &: \frac{A^2T}{3} + An_1\sqrt{\frac{T}{3}} > -An_2\sqrt{\frac{T}{3}} \rightarrow A\sqrt{T/3} + n_1 > -n_2
\end{aligned} \tag{1.13}$$

(1.13) is quite similar the case of Example 6.2 in the notes, “ECE376_Dimensionality of Signals_ASK_PSK_QAM_FSK_Jan 2013_HTE”. Hence Figs. 6.10, 6.11 and 6.12 will be valid, similarly (6.29) to (6.31) will be valid. Thus P_e will become

$$P_e = 1 - P_c = 1 - \frac{1}{(\pi N_0)^{0.5}} \int_{-\sqrt{\epsilon_s}}^{\infty} \exp\left(-\frac{n_1^2}{N_0}\right) dn_1 \frac{1}{(\pi N_0)^{0.5}} \int_{-(n_1 + \sqrt{\epsilon_s}) \tan\left(\frac{\pi}{4}\right)}^{(n_1 + \sqrt{\epsilon_s}) \tan\left(\frac{\pi}{4}\right)} \exp\left(-\frac{n_2^2}{N_0}\right) dn_2 \tag{1.14}$$

Fig. 1.3 clearly shows that there will be no difference in P_e (formulation) if another signal other than $s_1(t)$ were to be transmitted.

2. (30 Points) Answer the following questions as **True** or **False**. For the **False** ones give the correct answer or the reason. For the **True** ones, justify your answer
- a) During demodulation of AM, we use a local carrier at receiver which is phase synchronized to the transmitted carrier : True, since when phase difference between the transmitted carrier and the local one takes on values $\pi/2$, $3\pi/2$, then demodulated message signal goes to zero.

 - b) Under the conditions of equal energy, ASK has lower probability of error than PSK, QAM and FSK : False, at high M ary values, QAM is the best, then comes PSK and probably FSK, finally ASK has the highest probability of error.

 - c) Demodulation means extracting the carrier from the modulated signal : False, demodulation means extracting the message signal from the modulated signal (carrier).

 - d) PM has better noise performance than FM : On the contrary, FM has better noise performance, since there is bandwidth expansion in FM during modulation and the reverse during demodulation.

 - e) Correlation metrics evaluation makes a decision by measuring the distance between the received vector and all possible signals to be transmitted : False, it is the distance metrics evaluation that measures the distance between the received vector and all possible signals to be transmitted.

 - f) The number of correlation metrics evaluations depends on the dimensionality of the transmitted signal set : False it depends on the M ary value of the signal set.