Çankaya University – ECE Department – ECE 376 (MT)

Student Name : Date : 09.04.2013
Student Number : Open Source Exam

Questions

1. (70 Points) Constellations A and B are as illustrated in Figs. 1.1 and 1.2.

   a. Identify the type of modulation and dimensionality in these constellations. Write and plot the mathematical expression for the basis functions, $\psi_1^A(t)$, $\psi_2^A(t)$ and $\psi_1^B(t)$, $\psi_2^B(t)$, write for the signal vectors $\mathbf{s}_1^A \cdots \mathbf{s}_8^A$ and $\mathbf{s}_1^B \cdots \mathbf{s}_8^B$, write and plot the corresponding signal waveforms of $s_1^A(t) \cdots s_8^A(t)$ and $s_1^B(t) \cdots s_8^B(t)$. Find the distance between signal vector ends in both constellations. Determine what the vector lengths of A and B should be such that both constellation A and constellation B use the same total or average energy.
   b. Draw the demodulator as correlator and matched filter. Assuming that the signal $s_1^A(t)$ from constellation A and the signal $s_1^B(t)$ from constellation B are transmitted, find the outputs of the correlator and matched filter.
   c. Find the probability of error and decision regions via the evaluations of correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$ again assuming $s_1^A(t)$ from constellation A and $s_1^B(t)$ from constellation B were transmitted. Comment on whether you find any probability of error performance difference between constellation A and constellation B.

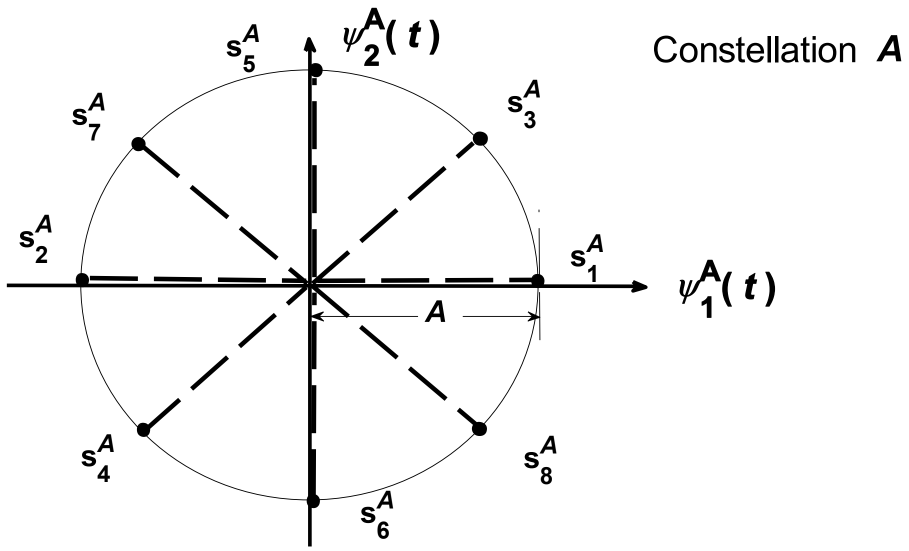

Fig. 1.1 Constellation A.

Note : You can write for the signal vectors $\mathbf{s}_1^A \cdots \mathbf{s}_8^A$ and $\mathbf{s}_1^B \cdots \mathbf{s}_8^B$ in time divided notation or in complex notation.

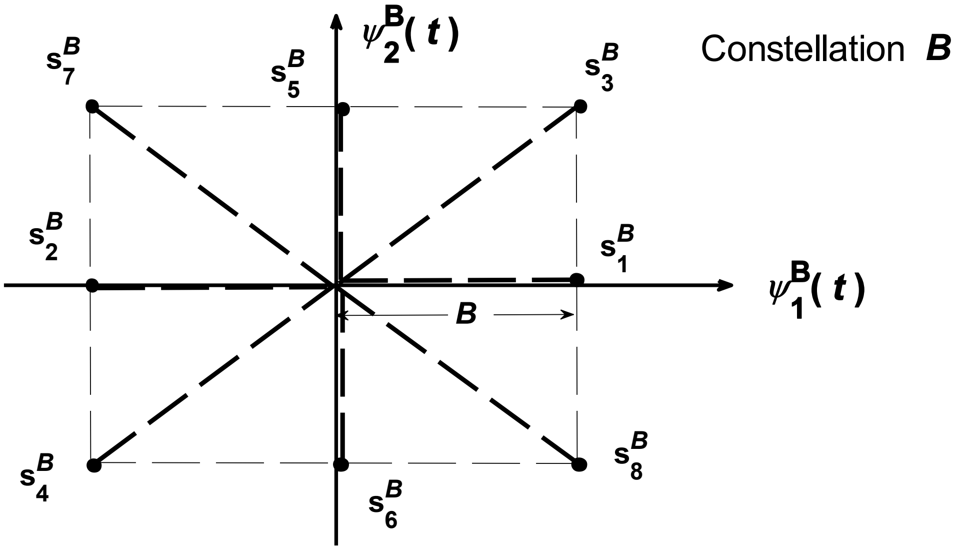

Fig. 1.2 Constellation B.

Solution : a. Constellation A is 8 PSK, while constellation B is 8 QAM. Orthonormalized basis functions, $\psi_1^A(t)$, $\psi_2^A(t)$ and $\psi_1^B(t)$, $\psi_2^B(t)$ are common in both cases as illustrated in Fig. 1.1 and (1.1) below.

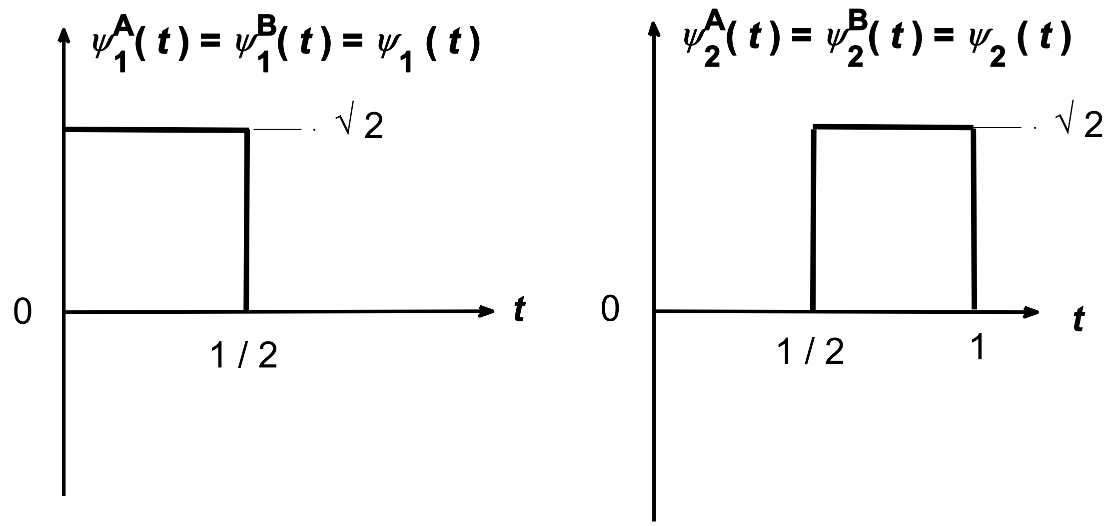

Fig. 1.3 The orthonormalized basis functions for Q1.

$$\psi_1^A(t) = \psi_1^B(t) = \psi_1(t) = \begin{cases} \sqrt{2} & 0 \le t \le 1/2 \\ 0 & \text{otherwise} \end{cases} \quad \psi_2^A(t) = \psi_2^B(t) = \psi_2(t) = \begin{cases} \sqrt{2} & 1/2 \le t \le 1 \\ 0 & \text{otherwise} \end{cases} \quad (1.1)$$

By using (1.1), Fig. 1.1 and Constellations A and B, we obtain the followings expressions for the signal vectors $\mathbf{s}_1^A \cdots \mathbf{s}_8^A$ and $\mathbf{s}_1^B \cdots \mathbf{s}_8^B$, the corresponding signal waveforms of $s_1^A(t) \cdots s_8^A(t)$ and $s_1^B(t) \cdots s_8^B(t)$.

$$s_1^A(t)=\begin{cases} A\sqrt{2} & 0\le t\le 1/2 \\ 0 & \text{otherwise} \end{cases},\ s_1^A(t)=A\psi_1(t),\ \mathbf{s_1^A}=[s_{11}^A,\ s_{12}^A]=[A,\ 0]=A\exp(j2\pi),\ \varepsilon_{s_1^A}=\left\|\mathbf{s_1^A}\right\|^2=A^2$$

$$s_3^A(t)=\begin{cases} A & 0\le t\le 1 \\ 0 & \text{otherwise} \end{cases},\ s_3^A(t)=\frac{A}{\sqrt{2}}\psi_1(t)+\frac{A}{\sqrt{2}}\psi_2(t)$$

$$\mathbf{s_3^A}=[s_{31}^A,\ s_{32}^A]=\left[\frac{A}{\sqrt{2}},\ \frac{A}{\sqrt{2}}\right]=A\exp(j\pi/4),\ \varepsilon_{s_3^A}=\left\|\mathbf{s_3^A}\right\|^2=A^2$$

$$s_5^A(t)=\begin{cases} A\sqrt{2} & 1/2\le t\le 1 \\ 0 & \text{otherwise} \end{cases},\ s_5^A(t)=A\psi_2(t),\ \mathbf{s_5^A}=[s_{51}^A,\ s_{52}^A]=[0,\ A]=A\exp(j\pi/2),\ \varepsilon_{s_5^A}=\left\|\mathbf{s_5^A}\right\|^2=A^2$$

$$s_7^A(t)=\begin{cases} -A & 0\le t\le 1/2 \\ A & 1/2\le t\le 1 \\ 0 & \text{otherwise} \end{cases},\ s_7^A(t)=\frac{-A}{\sqrt{2}}\psi_1(t)+\frac{A}{\sqrt{2}}\psi_2(t)$$

$$\mathbf{s_7^A}=[s_{71}^A,\ s_{72}^A]=\left[\frac{-A}{\sqrt{2}},\ \frac{A}{\sqrt{2}}\right]=A\exp(j3\pi/4),\ \varepsilon_{s_7^A}=\left\|\mathbf{s_7^A}\right\|^2=A^2$$

$$s_2^A(t)=\begin{cases} -A\sqrt{2} & 0\le t\le 1/2 \\ 0 & \text{otherwise} \end{cases},\ s_2^A(t)=-A\psi_1(t),\ \mathbf{s_2^A}=[s_{21}^A,\ s_{22}^A]=[-A,\ 0]=A\exp(j\pi),\ \varepsilon_{s_2^A}=\left\|\mathbf{s_2^A}\right\|^2=A^2$$

$$s_4^A(t)=\begin{cases} -A & 0\le t\le 1 \\ 0 & \text{otherwise} \end{cases},\ s_4^A(t)=\frac{-A}{\sqrt{2}}\psi_1(t)-\frac{A}{\sqrt{2}}\psi_2(t)$$

$$\mathbf{s_4^A}=[s_{41}^A,\ s_{42}^A]=\left[\frac{-A}{\sqrt{2}},\ \frac{-A}{\sqrt{2}}\right]=A\exp(j5\pi/4),\ \varepsilon_{s_4^A}=\left\|\mathbf{s_4^A}\right\|^2=A^2$$

$$s_6^A(t)=\begin{cases} -A\sqrt{2} & 1/2\le t\le 1 \\ 0 & \text{otherwise} \end{cases},\ s_6^A(t)=-A\psi_2(t),\ \mathbf{s_6^A}=[s_{61}^A,\ s_{62}^A]=[0,\ -A]=A\exp(j3\pi/2),\ \varepsilon_{s_6^A}=\left\|\mathbf{s_6^A}\right\|^2=A^2$$

$$s_8^A(t)=\begin{cases} A & 0\le t\le 1/2 \\ -A & 1/2\le t\le 1 \\ 0 & \text{otherwise} \end{cases},\ s_8^A(t)=\frac{A}{\sqrt{2}}\psi_1(t)-\frac{A}{\sqrt{2}}\psi_2(t)$$

$$\mathbf{s_8^A}=[s_{81}^A,\ s_{82}^A]=\left[\frac{A}{\sqrt{2}},\ \frac{-A}{\sqrt{2}}\right]=A\exp(j7\pi/4),\ \varepsilon_{s_8^A}=\left\|\mathbf{s_8^A}\right\|^2=A^2$$

$$\varepsilon_{s_1^A}=\varepsilon_{s_2^A}=\varepsilon_{s_3^A}=\varepsilon_{s_4^A}=\varepsilon_{s_5^A}=\varepsilon_{s_6^A}=\varepsilon_{s_7^A}=\varepsilon_{s_8^A}=\varepsilon_{s^A}=A^2$$

$$d_{s_1^A s_3^A}^2\,(\text{minimum distance})=A^2\sqrt{2}\left(\sqrt{2}-1\right) \qquad (1.2)$$

$$s_1^B(t)=\begin{cases} B\sqrt{2} & 0\le t\le 1/2 \\ 0 & \text{otherwise}\end{cases}\ ,\ s_1^B(t)=B\psi_1(t)\ ,\ \mathbf{s_1^B}=\left[s_{11}^B,\ s_{12}^B\right]=[B,\ 0]=B\exp(j2\pi)\ ,\ \varepsilon_{s_1^B}=\left\|\mathbf{s_1^B}\right\|^2=B^2$$

$$s_3^B(t)=\begin{cases} B\sqrt{2} & 0\le t\le 1 \\ 0 & \text{otherwise}\end{cases}\ ,\ s_3^B(t)=B\psi_1(t)+B\psi_2(t)$$

$$\mathbf{s_3^B}=\left[s_{31}^B,\ s_{32}^B\right]=[B,\ B]=B\sqrt{2}\exp(j\pi/4)\ ,\ \varepsilon_{s_3^B}=\left\|\mathbf{s_3^B}\right\|^2=2B^2$$

$$s_5^B(t)=\begin{cases} B\sqrt{2} & 1/2\le t\le 1 \\ 0 & \text{otherwise}\end{cases}\ ,\ s_5^B(t)=B\psi_2(t)\ ,\ \mathbf{s_5^B}=\left[s_{51}^B,\ s_{52}^B\right]=[0,\ B]=B\exp(j\pi/2)\ ,\ \varepsilon_{s_5^B}=\left\|\mathbf{s_5^B}\right\|^2=B^2$$

$$s_7^B(t)=\begin{cases} -B & 0\le t\le 1/2 \\ B & 1/2\le t\le 1 \\ 0 & \text{otherwise}\end{cases}\ ,\ s_7^B(t)=-B\psi_1(t)+B\psi_2(t)$$

$$\mathbf{s_7^B}=\left[s_{71}^B,\ s_{72}^B\right]=[-B,\ B]=B\sqrt{2}\exp(j3\pi/4)\ ,\ \varepsilon_{s_7^B}=\left\|\mathbf{s_7^B}\right\|^2=2B^2$$

$$s_2^B(t)=\begin{cases} -B\sqrt{2} & 0\le t\le 1/2 \\ 0 & \text{otherwise}\end{cases}\ ,\ s_2^B(t)=-B\psi_1(t)\ ,\ \mathbf{s_2^B}=\left[s_{21}^B,\ s_{22}^B\right]=[-B,\ 0]=B\exp(j\pi)\ ,\ \varepsilon_{s_2^B}=\left\|\mathbf{s_2^B}\right\|^2=B^2$$

$$s_4^B(t)=\begin{cases} -B\sqrt{2} & 0\le t\le 1 \\ 0 & \text{otherwise}\end{cases}\ ,\ s_4^B(t)=-B\psi_1(t)-B\psi_2(t)$$

$$\mathbf{s_4^B}=\left[s_{41}^B,\ s_{42}^B\right]=[-B,\ -B]=B\sqrt{2}\exp(j5\pi/4)\ ,\ \varepsilon_{s_4^B}=\left\|\mathbf{s_4^B}\right\|^2=2B^2$$

$$s_6^B(t)=\begin{cases} -B\sqrt{2} & 1/2\le t\le 1 \\ 0 & \text{otherwise}\end{cases}\ ,\ s_6^B(t)=-B\psi_2(t)\ ,\ \mathbf{s_6^B}=\left[s_{61}^B,\ s_{62}^B\right]=[0,\ -B]=B\exp(j3\pi/2)\ ,\ \varepsilon_{s_6^A}=\left\|\mathbf{s_6^A}\right\|^2=B^2$$

$$s_8^B(t)=\begin{cases} B\sqrt{2} & 0\le t\le 1/2 \\ -B\sqrt{2} & 1/2\le t\le 1 \\ 0 & \text{otherwise}\end{cases}\ ,\ s_8^B(t)=B\psi_1(t)-B\psi_2(t)$$

$$\mathbf{s_8^B}=\left[s_{81}^B,\ s_{82}^B\right]=[B,\ -B]=B\sqrt{2}\exp(j7\pi/4)\ ,\ \varepsilon_{s_8^B}=\left\|\mathbf{s_8^B}\right\|^2=2B^2$$

$$\varepsilon_{s_1^B}=\varepsilon_{s_2^B}=\varepsilon_{s_5^B}=\varepsilon_{s_6^B}=B^2\quad ,\quad \varepsilon_{s_3^B}=\varepsilon_{s_4^B}=\varepsilon_{s_7^B}=\varepsilon_{s_8^B}=2B^2$$

$$d_{s_1^B s_3^B}^2\,(\text{minimum distance})=B^2 \qquad (1.3)$$

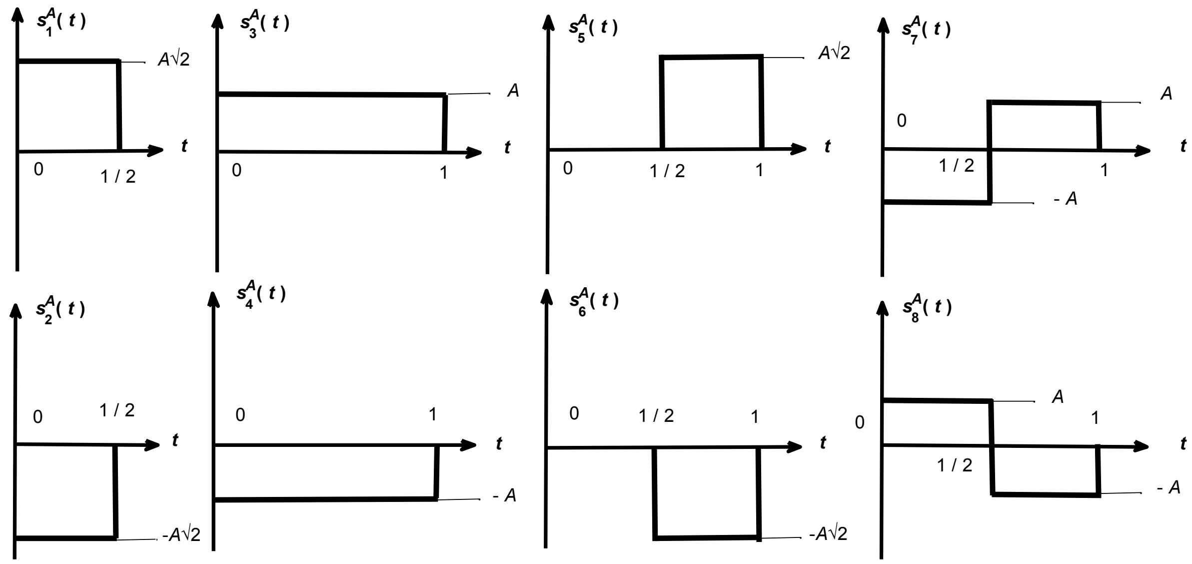

Fig. 1.4 Time waveforms of $s_1^A(t)\cdots s_8^A(t)$ for constellation A.

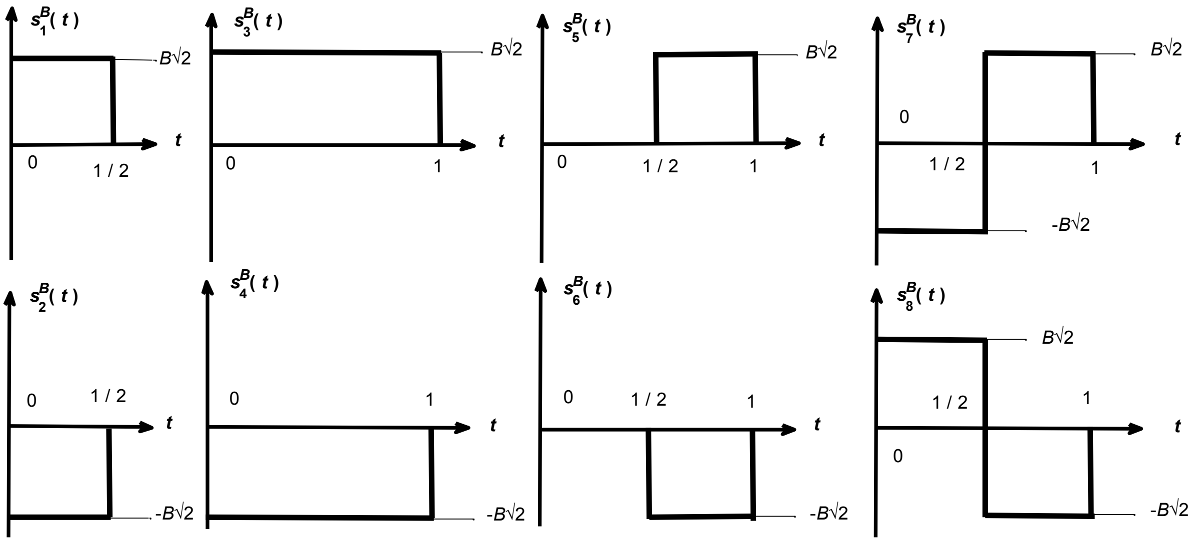

Fig. 1.5 Time waveforms of $s_1^B(t) \cdots s_8^B(t)$ for constellation B.

For the total (or average) energies to be equal for constellation A and B, we should have

$$8A^2 = 4B^2 + 4 \times 2B^2 \rightarrow A = (1.5)^{0.5} B \quad (1.4)$$

b. For both constellations, the relevant correlator and matched filter block diagrams are given Figs. 6.7a and b in ECE376_ Dimensionality of Signals_ASK_PSK_QAM_FSK_Jan 2013_HTE, so they are not repeated here.

The outputs in the upper and lower arms of either correlator and matched filter if $s_1^A(t)$ from constellation A and the signal $s_1^B(t)$ from constellation B are transmitted, will be

$$y_{1A} = r_{1A} = \int_0^{0.5} r(t)\ \psi_1(t)\ dt = \int_0^{0.5} s_1^A(t)\ \psi_1(t)\ dt + \int_0^{0.5} n(t)\ \psi_1(t)\ dt$$

$$= A + n_1 \quad , \quad n_1 = \int_0^{0.5} n(t)\ \psi_1(t)\ dt$$

$$y_{2A} = r_{2A} = \int_0^{0.5} r(t)\ \psi_2(t)\ dt = \int_0^{0.5} s_1^A(t)\ \psi_2(t)\ dt + \int_0^{0.5} n(t)\ \psi_2(t)\ dt$$

$$= 0 + n_2 \quad , \quad n_2 = \int_0^{0.5} n(t)\ \psi_2(t)\ dt \quad , \quad \mathbf{r_A} = [r_{1A};\ r_{2A}] = \begin{bmatrix} A + n_1 \\ n_2 \end{bmatrix} \quad (1.5)$$

$$y_{1B} = r_{1B} = \int_0^{0.5} r(t)\ \psi_1(t)\ dt = \int_0^{0.5} s_1^B(t)\ \psi_1(t)\ dt + \int_0^{0.5} n(t)\ \psi_1(t)\ dt$$

$$= B + n_1 \quad , \quad n_1 = \int_0^{0.5} n(t)\ \psi_1(t)\ dt$$

$$y_{2B} = r_{2B} = \int_0^{0.5} r(t)\ \psi_2(t)\ dt = \int_0^{0.5} s_1^B(t)\ \psi_2(t)\ dt + \int_0^{0.5} n(t)\ \psi_2(t)\ dt$$

$$= 0 + n_2 \quad , \quad n_2 = \int_0^{0.5} n(t)\ \psi_2(t)\ dt \quad , \quad \mathbf{r_B} = [r_{1B};\ r_{2B}] = \begin{bmatrix} B + n_1 \\ n_2 \end{bmatrix} \tag{1.6}$$

For the cases given (1.5) and (1.6), we should theoretically evaluate $C(\mathbf{r}, \mathbf{s}_m)$ for $m = 1 \cdots 8$, as shown in Example 6.2 of Dimensionality of Signals_ASK_PSK_QAM_FSK_Jan 2013_HTE, it is sufficient to find $C(\mathbf{r_A}, \mathbf{s_1^A})$, $C(\mathbf{r_A}, \mathbf{s_3^A})$ and $C(\mathbf{r_B}, \mathbf{s_1^B})$, $C(\mathbf{r_B}, \mathbf{s_3^B})$ instead. These are given below.

$$C(\mathbf{r_A}, \mathbf{s_1^A}) = 2\,\mathbf{s_1^A} \cdot \mathbf{r_A} - \left\|\mathbf{s_1^A}\right\|^2 = 2[A,\ 0]\begin{bmatrix} A + n_1 \\ n_2 \end{bmatrix} - A^2 = A^2 + 2An_1$$

$$C(\mathbf{r_A}, \mathbf{s_3^A}) = 2\,\mathbf{s_3^A} \cdot \mathbf{r_A} - \left\|\mathbf{s_3^A}\right\|^2 = 2\begin{bmatrix} \frac{A}{\sqrt{2}}, & \frac{A}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} A + n_1 \\ n_2 \end{bmatrix} - A^2 = \left(\sqrt{2} - 1\right)A^2 + \sqrt{2}An_1 + \sqrt{2}An_2$$

$$C(\mathbf{r_B}, \mathbf{s_1^B}) = 2\,\mathbf{s_1^B} \cdot \mathbf{r_B} - \left\|\mathbf{s_1^B}\right\|^2 = 2[B,\ 0]\begin{bmatrix} B + n_1 \\ n_2 \end{bmatrix} - B^2 = B^2 + 2Bn_1$$

$$C(\mathbf{r_B}, \mathbf{s_3^B}) = 2\,\mathbf{s_3^A} \cdot \mathbf{r_B} - \left\|\mathbf{s_3^B}\right\|^2 = 2[B,\ B]\begin{bmatrix} B + n_1 \\ n_2 \end{bmatrix} - 2B^2 = 2Bn_1 + 2Bn_2 \tag{1.7}$$

Now for constellation A, impose the condition of correct decision to get

$$C(\mathbf{r_A}, \mathbf{s_1^A}) > C(\mathbf{r_A}, \mathbf{s_3^A}) \rightarrow C(\mathbf{r_A}, \mathbf{s_1^A}) - C(\mathbf{r_A}, \mathbf{s_3^A}) > 0$$

$$\overbrace{A^2 + 2An_1}^{C(\mathbf{r_A}, \mathbf{s_1^A})} - \overbrace{\left(\sqrt{2} - 1\right)A^2 + \sqrt{2}An_1 + \sqrt{2}An_2}^{C(\mathbf{r_A}, \mathbf{s_3^A})} > 0$$

$$\left(\sqrt{2} - 1\right)(A + n_1) > n_2 \ \text{ or } \ A + n_1 > \frac{n_2}{\sqrt{2} - 1} \ \text{ or } \ \frac{n_2}{A + n_1} < \sqrt{2} - 1 \tag{1.8}$$

The last line of (1.8) describes a straight line with an inclination of $\pi/8 \rightarrow 22.5^0$ to the horizontal axis. Taking into account the conditions arising from the other the correlation metrics, i.e.,

$$C(\mathbf{r_A}, \mathbf{s_1^A}) > C(\mathbf{r_A}, \mathbf{s_2^A}),\ C(\mathbf{r_A}, \mathbf{s_1^A}) > C(\mathbf{r_A}, \mathbf{s_4^A}),\ C(\mathbf{r_A}, \mathbf{s_1^A}) > C(\mathbf{r_A}, \mathbf{s_5^A})$$
$$C(\mathbf{r_A}, \mathbf{s_1^A}) > C(\mathbf{r_A}, \mathbf{s_6^A}),\ C(\mathbf{r_A}, \mathbf{s_1^A}) > C(\mathbf{r_A}, \mathbf{s_7^A}),\ C(\mathbf{r_A}, \mathbf{s_1^A}) > C(\mathbf{r_A}, \mathbf{s_8^A}) \tag{1.9}$$

We obtain the following

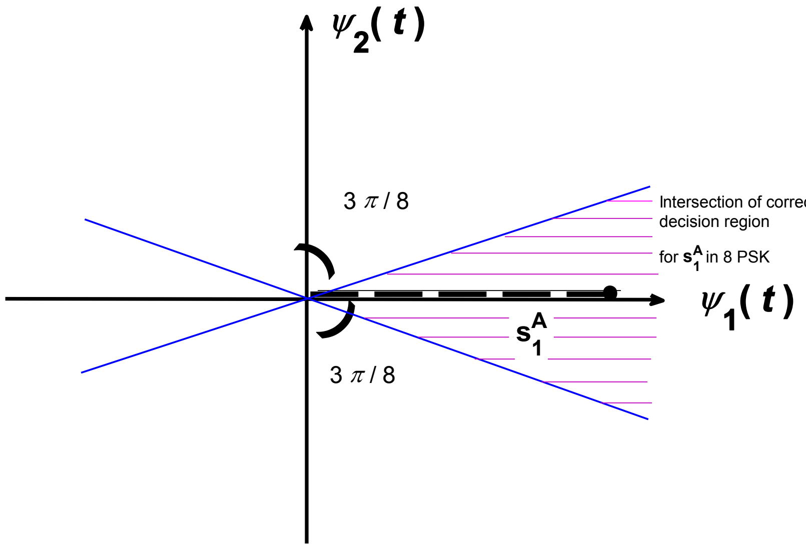

Fig. 1.6 Correct decision region for constellation A, when $s_1^A(t)$ is transmitted.

Based on (6.29) of Dimensionality of Signals_ASK_PSK_QAM_FSK_Jan 2013_HTE, we derive the probability error for the case of Fig. 1.6 as

$$P_e = 1 - \frac{1}{(\pi N_0)^{0.5}} \int_{-A}^{\infty} \exp\left(-\frac{n_1^2}{N_0}\right) dn_1 \frac{1}{(\pi N_0)^{0.5}} \int_{-(\sqrt{2}-1)(n_1+A)}^{(\sqrt{2}-1)(n_1+A)} \exp\left(-\frac{n_2^2}{N_0}\right) dn_2 \qquad (1.10)$$

Now for constellation B, again impose the condition of correct decision to get

$$C(\mathbf{r}_B, \mathbf{s}_1^B) > C(\mathbf{r}_B, \mathbf{s}_3^B) \rightarrow C(\mathbf{r}_B, \mathbf{s}_1^B) - C(\mathbf{r}_B, \mathbf{s}_3^B) > 0$$
$$\overbrace{B^2 + 2Bn_1}^{C(\mathbf{r}_B, \mathbf{s}_1^B)} - \overbrace{2Bn_1 + 2Bn_2}^{C(\mathbf{r}_B, \mathbf{s}_3^B)} > 0 \rightarrow n_2 < 0.5B \qquad (1.11)$$

Together with the conditions arising from the other the correlation metrics, i.e.,

$$C(\mathbf{r}_b, \mathbf{s}_1^B) > C(\mathbf{r}_B, \mathbf{s}_2^B),\ C(\mathbf{r}_B, \mathbf{s}_1^B) > C(\mathbf{r}_B, \mathbf{s}_4^B),\ C(\mathbf{r}_B, \mathbf{s}_1^B) > C(\mathbf{r}_B, \mathbf{s}_5^B)$$
$$C(\mathbf{r}_B, \mathbf{s}_1^B) > C(\mathbf{r}_B, \mathbf{s}_6^B),\ C(\mathbf{r}_B, \mathbf{s}_1^B) > C(\mathbf{r}_B, \mathbf{s}_7^B),\ C(\mathbf{r}_B, \mathbf{s}_1^B) > C(\mathbf{r}_B, \mathbf{s}_8^B) \qquad (1.12)$$

We obtain the following graph

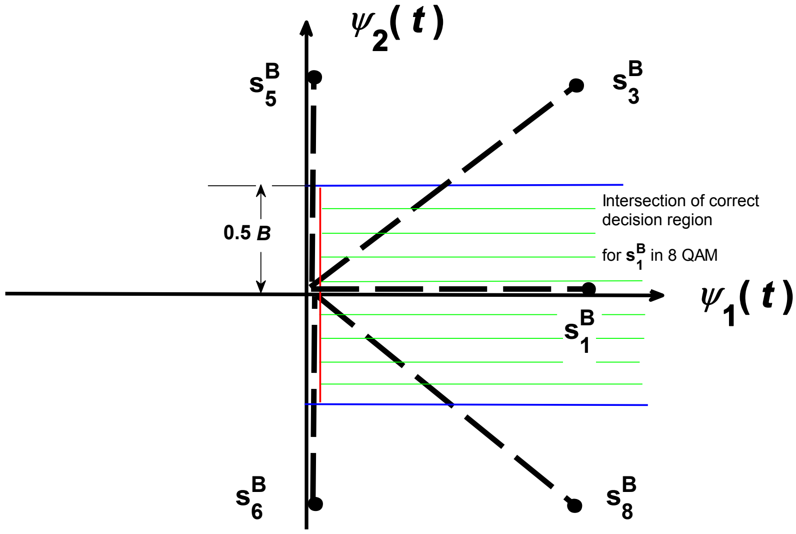

Fig. 1.7 Correct decision region for constellation B, when $s_1^B(t)$ is transmitted.

Finally we get the probability error for the case of Fig. 1.7 as

$$P_e = 1 - \frac{1}{(\pi N_0)^{0.5}} \int_0^{\infty} \exp\left(-\frac{n_1^2}{N_0}\right) dn_1 \frac{1}{(\pi N_0)^{0.5}} \int_{-0.5B}^{0.5B} \exp\left(-\frac{n_2^2}{N_0}\right) dn_2 \tag{1.13}$$

Comparison of (1.10) with (1.13) and finding which one offers lower probability of error (under the condition of using the same energy) is left as exercise.

2. (30 Points) Answer the following questions as **True** or **False.** For the **False** ones give the correct answer or the reason. For the **True** ones, justify your answer

a) AM offers SNR improvement during demodulation process : False, it is FM that offers SNR improvement during demodulation process, not AM.

b) The role of the matched filter is to invert the input signal : False, matched filter passes the input signal through a response matched to the input itself.

c) In ASK, symbol duration is reduced as M increases : False, reverse is true, but this is not specific to ASK, the same applies to PSK, QAM and FSK, provided that (binary) input rate, thus the bit duration, is kept constant.

d) PLL is used in the demodulation of FM signal : True, phase locked loop (PLL) is one possible modulators of FM signal as illustrated in ECE 376_AM_FM Demodulation_Jan 2013_HTE.

e) Optimum detector uses an optimum signal input : False, optimum detector tries to seek the nearest match of the transmitted signal by maximizing
$P\left(\text{signal } \mathbf{s}_m \text{was transmitted} \middle| \text{given received signal vector } \mathbf{r}\right) = P\left(\mathbf{s}_m \middle| \mathbf{r}\right)$

f) For a given modulating message signal, FM and PM use the same bandwidth : False, generally FM uses higher bandwidth, which is used as an advantage of SNR improvement during demodulation process.